Elementary Graph Algorithms

Graphs

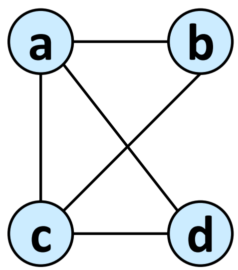
- *Graph $G = (V, E)$*
  - V = set of vertices
  - E = set of edges $\subseteq (V \times V)$
- Types of graphs
  - Undirected: edge $(u, v) = (v, u)$; for all v, $(v, v) \notin E$ (No self loops.)
  - Directed: (u, v) is edge from u to v, denoted as $u \rightarrow v$. Self loops are allowed.
  - Weighted: each edge has an associated weight, given by a weight function $w : E \rightarrow \mathbf{R}$.
  - Dense: $|E| \approx |V|^2$.
  - Sparse: $|E| << |V|^2$.
- $|E| = O(|V|^2)$

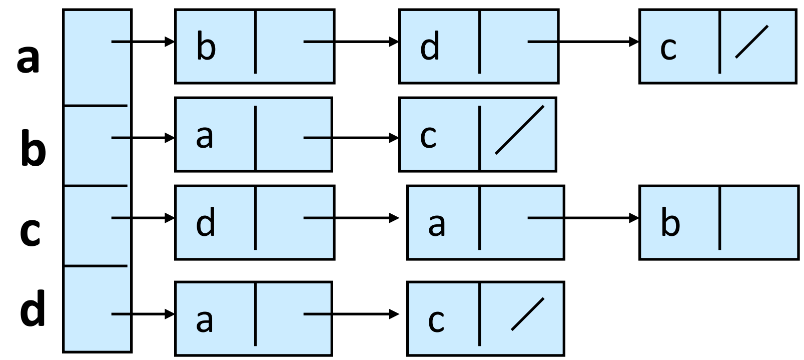
Graphs

- If $(u, v) \in E$, then vertex v is adjacent to vertex u.
- Adjacency relationship is:
  - Symmetric if G is undirected.
  - Not necessarily so if G is directed.
- If G is connected:
  - There is a path between every pair of vertices.
  - $|E| \geq |V| - 1$.
  - Furthermore, if $|E| = |V| - 1$, then G is a tree.

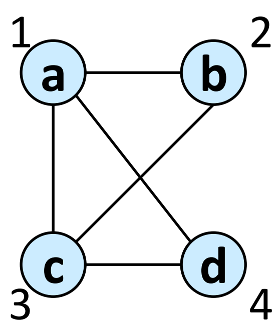
Representation of Graphs

- Two standard ways.
  - Adjacency Lists.





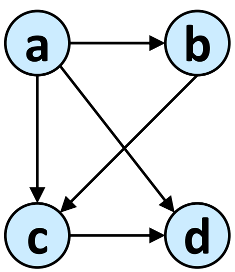
  - Adjacency Matrix.

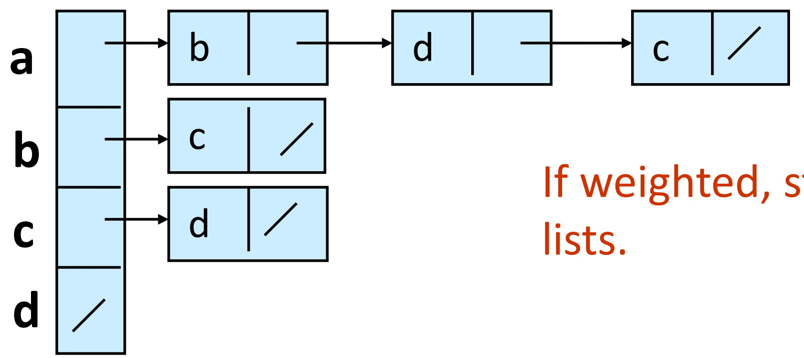


	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

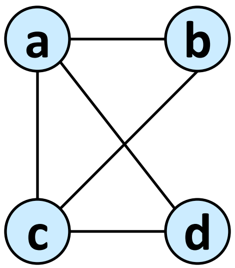
Adjacency Lists

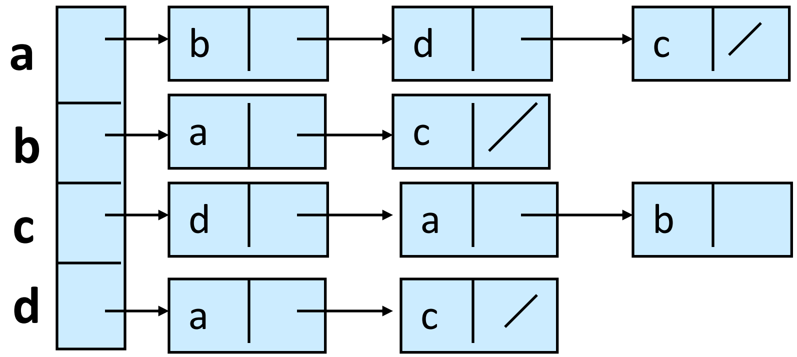
- Consists of an array *Adj* of $|V|$ lists.
- One list per vertex.
- For $u \in V$, $Adj[u]$ consists of all vertices adjacent to u.





If weighted, store weights also in adjacency lists.





Storage Requirement

- For directed graphs:
  - Sum of lengths of all adj. lists is

    $$\sum_{v \in V} \text{out-degree}(v) = |E|$$

    No. of edges leaving v
  - Total storage: $\Theta(V+E)$
- For undirected graphs:
  - Sum of lengths of all adj. lists is

    $$\sum_{v \in V} \text{degree}(v) = 2|E|$$

    No. of edges incident on v. Edge (u,v) is incident on vertices u and v.
  - Total storage: $\Theta(V+E)$

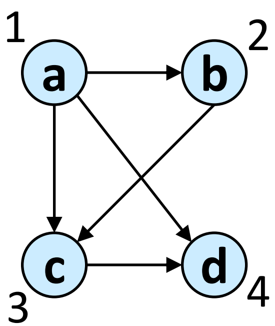
Pros and Cons: adj list

- Pros
  - Space-efficient, when a graph is sparse.
- Cons
  - Determining if an edge $(u,v) \in G$ is not efficient.
    - Have to search in u's adjacency list. $\Theta(\text{degree}(u))$ time.

Adjacency Matrix

- $|V| \times |V|$ matrix A.
- Number vertices from 1 to $|V|$ in some arbitrary manner.
- A is then given by:

$$A[i, j] = a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$



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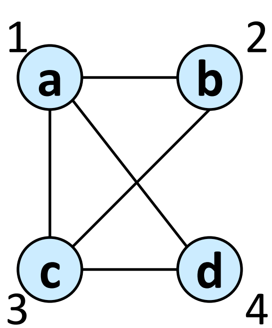
4	0	0	0	0



	1	2	3	4
1	0	1	1	1
2	1	0	1	0
3	1	1	0	1
4	1	0	1	0

$A = A^{\mathrm{T}}$ for undirected graphs.

Space and Time

- **Space:** $\Theta(V^2)$.
  - Not memory efficient for large graphs.
- **Time:** to list all vertices adjacent to u: $\Theta(V)$.
- **Time:** to determine if $(u, v) \in E$: $\Theta(1)$.
- Can store weights instead of bits for weighted graph.

Graph-searching Algorithms

- Searching a graph:
  - Systematically follow the edges of a graph to visit the vertices of the graph.
- Used to discover the structure of a graph.
- Standard graph-searching algorithms.
  - Breadth-first Search (BFS).
  - Depth-first Search (DFS).

Breadth-first Search

- **Input:** Graph $G = (V, E)$, either directed or undirected, and ***source vertex*** $s \in V$.
- **Output:**
  - $d[v]$ = distance (smallest # of edges, or shortest path) from s to v, for all $v \in V$. $d[v] = \infty$ if v is not reachable from s.
  - $\pi[v]$ = u such that (u, v) is last edge on shortest path $s \rightsquigarrow v$.
    - u is v's predecessor.
  - Builds breadth-first tree with root s that contains all reachable vertices.

Definitions:

Path between vertices u and v: Sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $u=v_1$ and $v = v_k$, and $(v_i, v_{i+1}) \in E$, for all $1 \leq i \leq k-1$.

Length of the path: Number of edges in the path.

Path is simple if no vertex is repeated.

Breadth-first Search

- Expands the frontier between discovered and undiscovered vertices uniformly across the breadth of the frontier.
  - A vertex is "discovered" the first time it is encountered during the search.
  - A vertex is "finished" if all vertices adjacent to it have been discovered.
- Colors the vertices to keep track of progress.
  - White – Undiscovered.
  - Gray – Discovered but not finished.
  - Black – Finished.
    - Colors are required only to reason about the algorithm. Can be implemented without colors.

BFS(G,s)

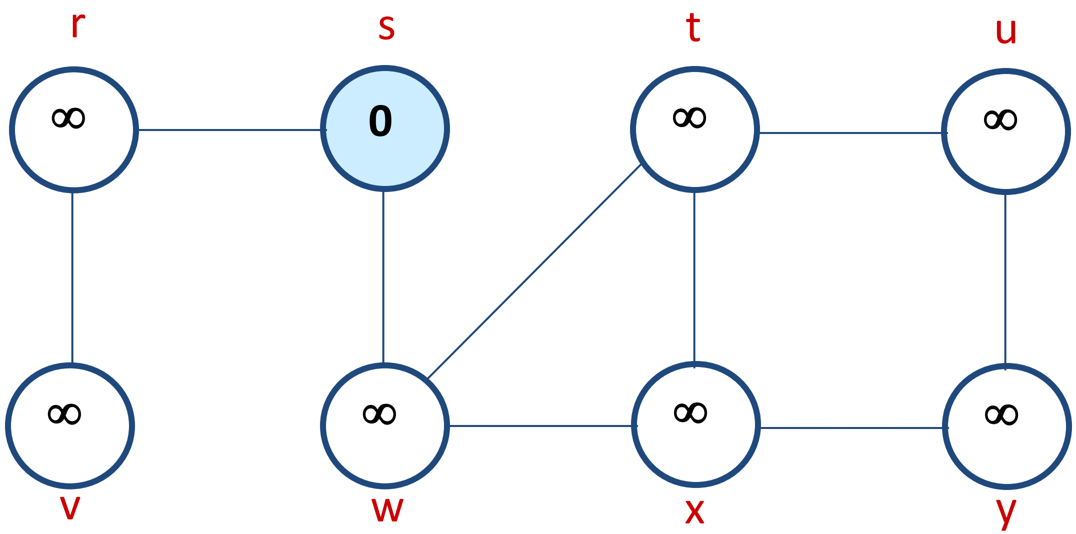
```
for each vertex u in V[G] – {s}
        do color[u] ← white
            d[u] ← ∝
            π[u] ← nil
color[s] ← gray
d[s] ← 0
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
        do u ← dequeue(Q)
                for each v in Adj[u]
                        do if color[v] = white
                                then color[v] ← gray
                                    d[v] ← d[u] + 1
                                    π[v] ← u
                                    enqueue(Q,v)
                color[u] ← black
```

white: undiscovered
gray: discovered
black: finished

Q: a queue of discovered vertices
color[*v*]: color of v
d[*v*]: distance from s to v
π[*u*]: predecessor of v

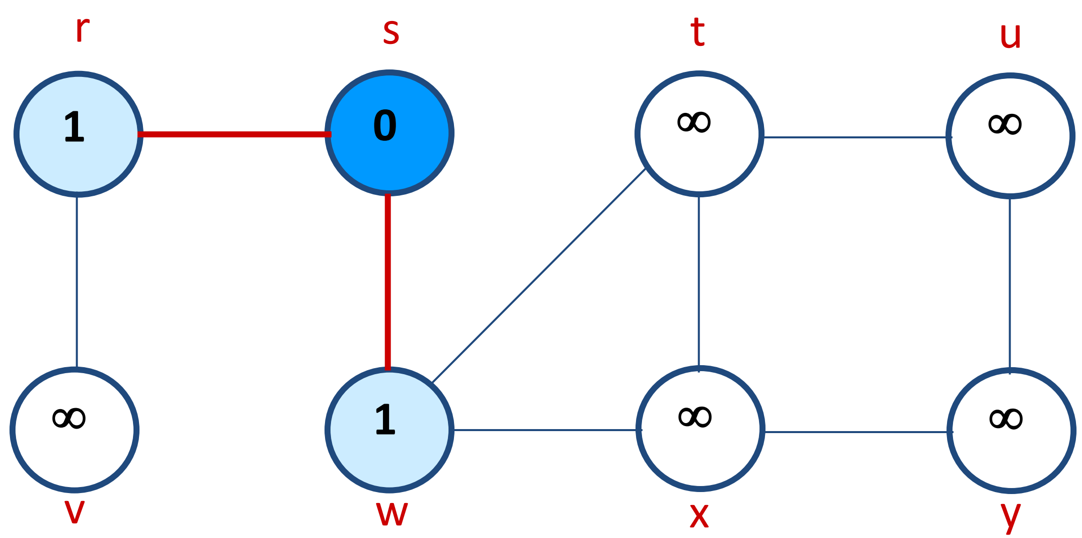
Example: animation.

Example (BFS)



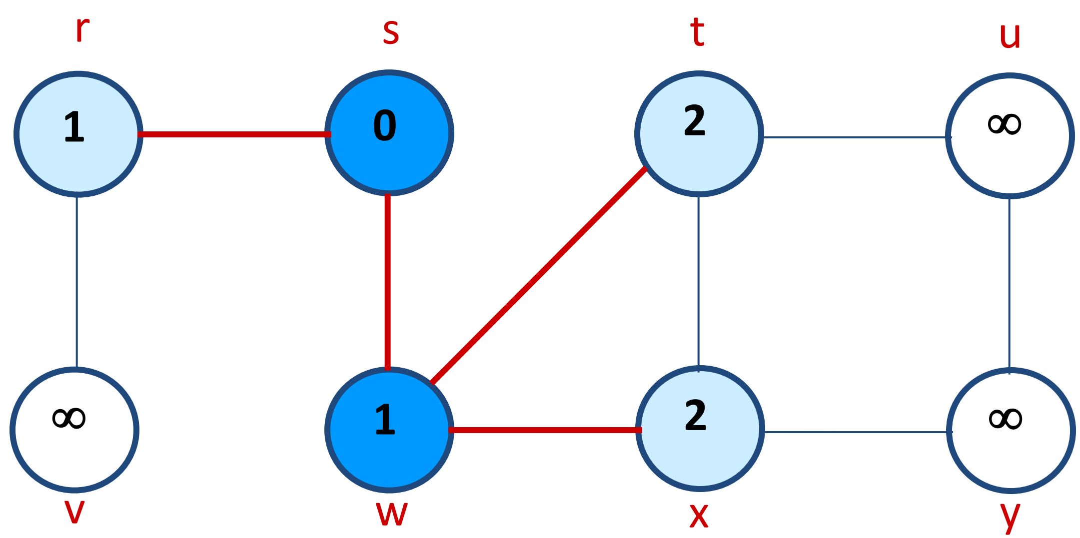
Q: s
0

Example (BFS)



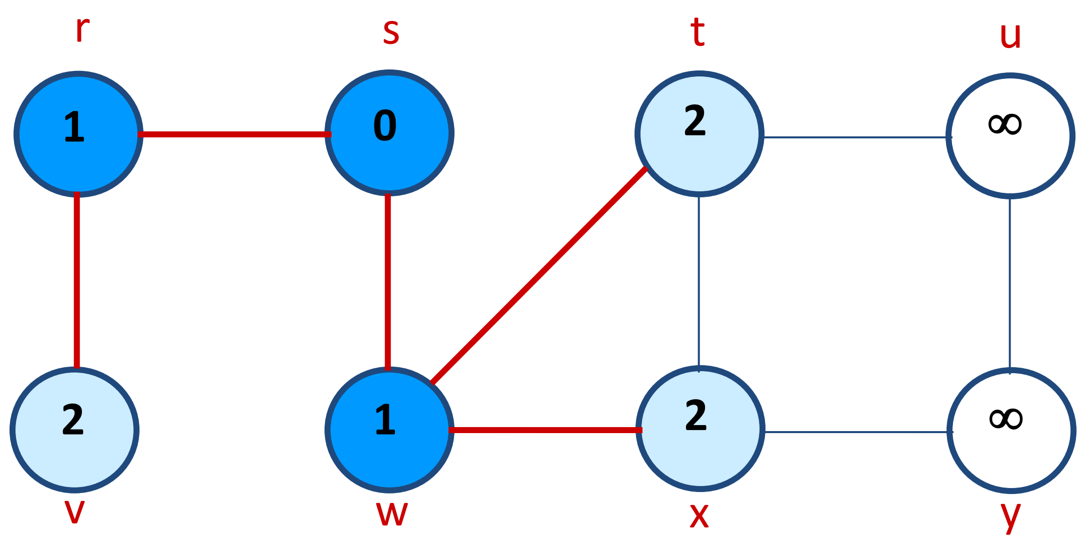
Q: w r
1 1

Example (BFS)



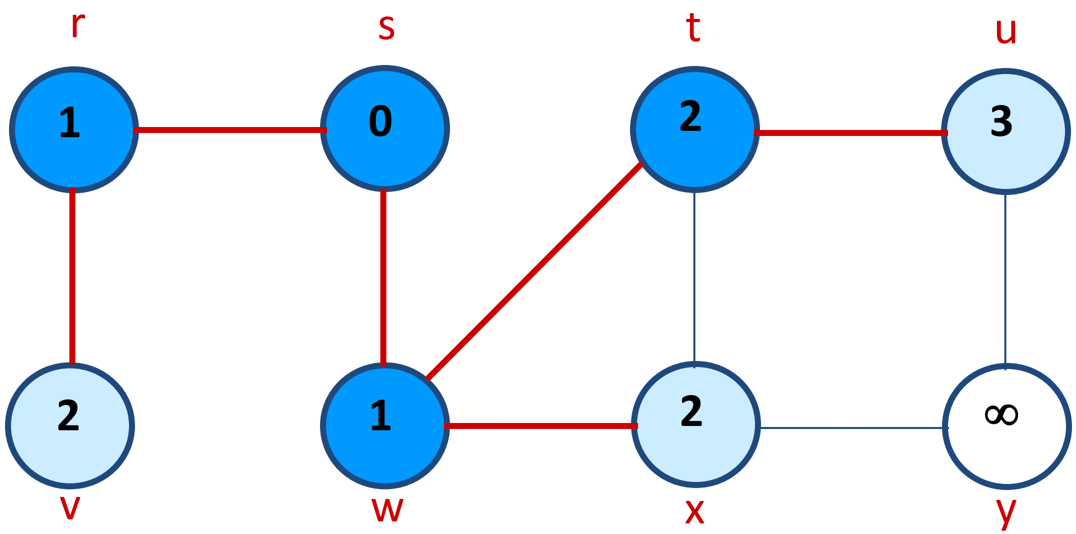
Q: r t x
1 2 2

Example (BFS)



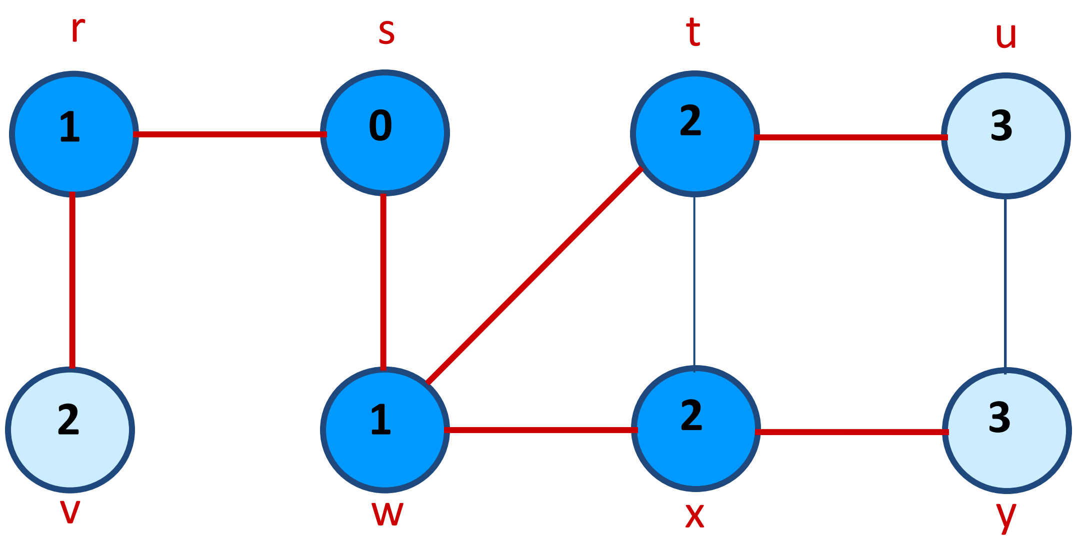
Q: t x v
2 2 2

Example (BFS)



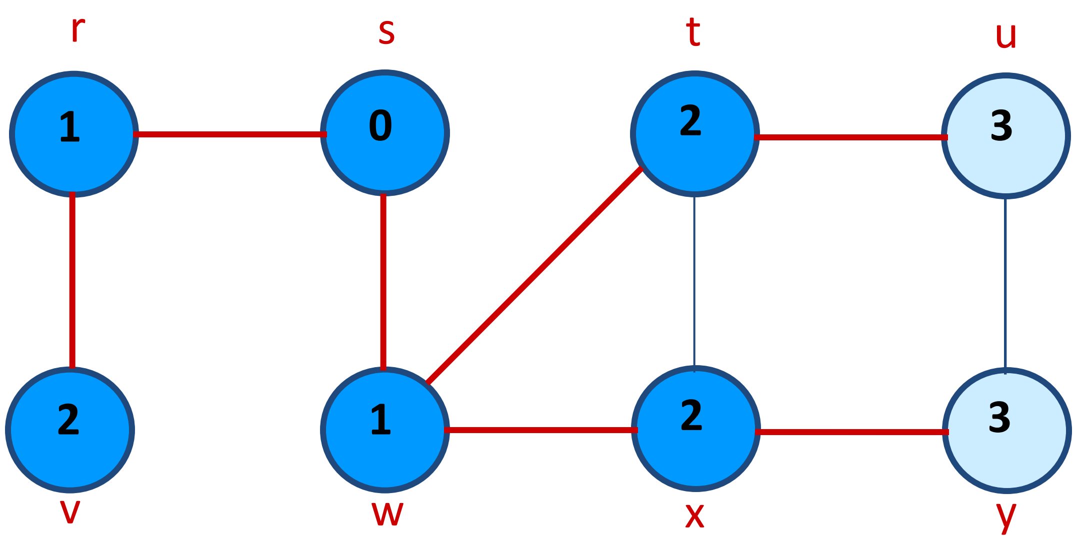
Q:	x	v	u
	2	2	3

Example (BFS)



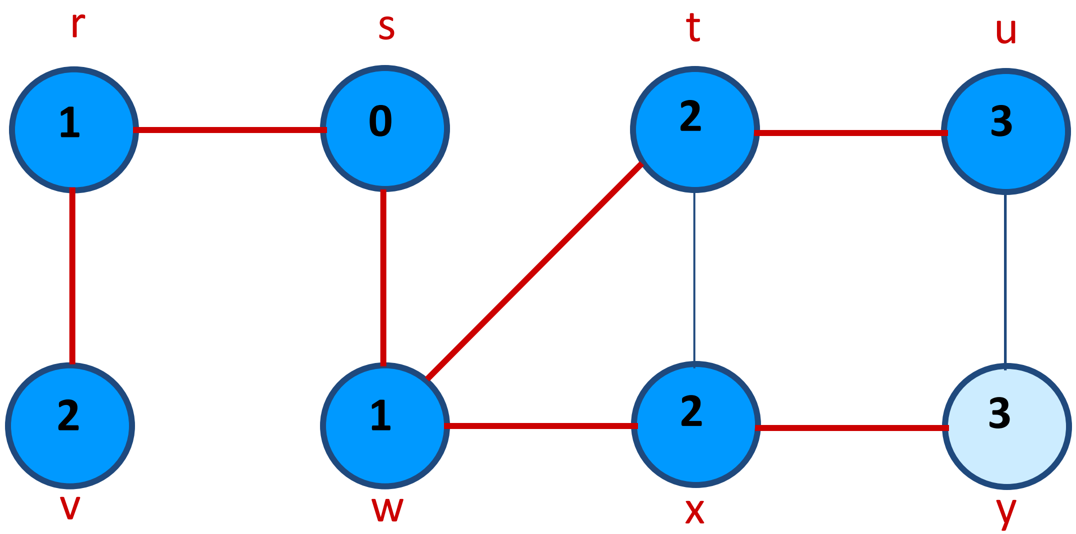
Q: v u y
2 3 3

Example (BFS)



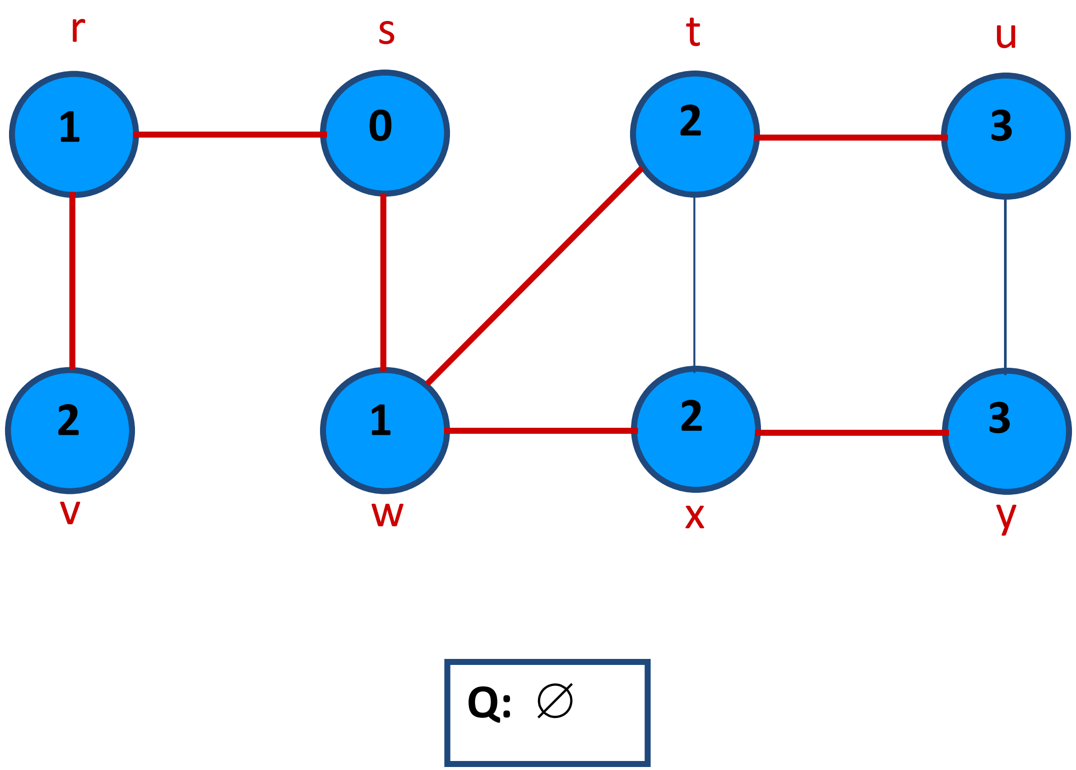
Q:	u	y
	3	3

Example (BFS)

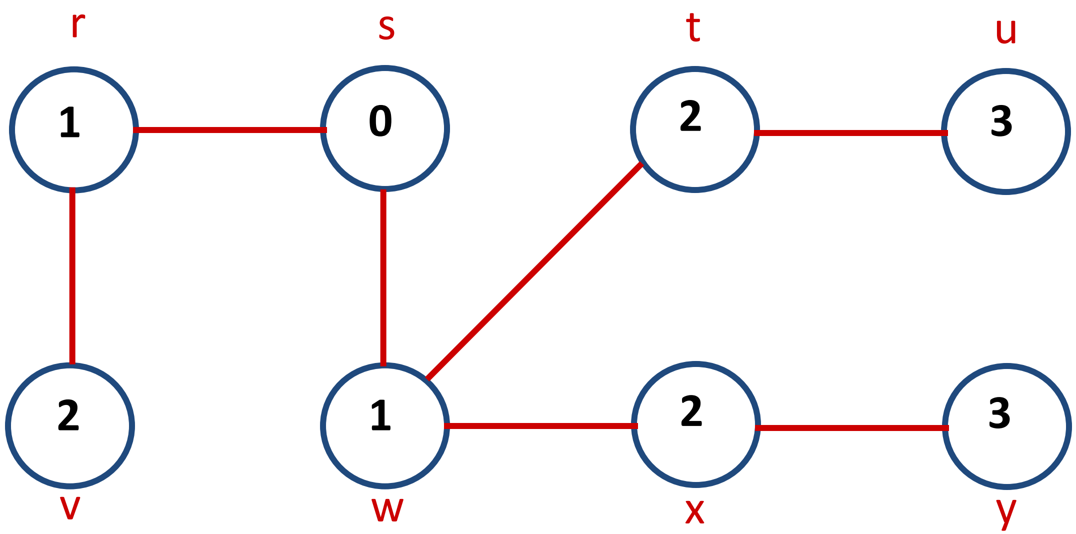


Q: y
3

Example (BFS)



Example (BFS)



BF Tree

Analysis of BFS

- Initialization takes $O(V)$.
- Traversal Loop
  - After initialization, each vertex is enqueued and dequeued at most once, and each operation takes $O(1)$. So, total time for queuing is $O(V)$.
  - The adjacency list of each vertex is scanned at most once. The sum of lengths of all adjacency lists is $\Theta(E)$.
- Summing up over all vertices => total running time of BFS is $O(V+E)$, linear in the size of the adjacency list representation of graph.
- .

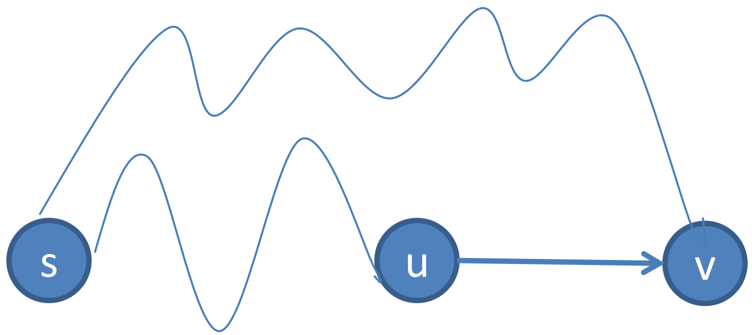
BFS(G,s)

```
for each vertex u in V[G] – {s}
        do color[u] ← white
            d[u] ← ∝
            π[u] ← nil
color[s] ← gray
d[s] ← 0
π[s] ← nil
Q ← Φ
enqueue(Q,s)
while Q ≠ Φ
        do u ← dequeue(Q)
                for each v in Adj[u]
                        do if color[v] = white
                            then color[v] ← gray
                                d[v] ← d[u] + 1
                                π[v] ← u
                                enqueue(Q,v)
                color[u] ← black
```

Lemma 1

Let $G = (V, E)$ be a directed or undirected graph, and let $s \in V$ be an arbitrary vertex. Then, for any edge $(u, v) \in E$,

$$\delta(s, v) \le \delta(s, u) + 1 .$$



Lemma 2

Let G = (V,E) be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex s ∈ V . Then upon termination, for each vertex ∈ V , the value v.*d* computed by BFS satisfies v.*d* *>=δ(s,v)*.

```
BFS(G,s)
for each vertex u in V[G] – {s}
        do color[u] ← white
           d[u] ← ∝
           π[u] ← nil
color[s] ← gray
d[s] ← 0
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enqueue(Q,s)
while Q ≠ Φ
        do u ← dequeue(Q)
                for each v in Adj[u]
                        do if color[v] = white
                                then color[v] ← gray
                                     d[v] ← d[u] + 1
                                     π[v] ← u
                                     enqueue(Q,v)
                color[u] ← black
```

- For initial case,
  - $\delta(s,s)=0$
  - $s.d=0$
  - $s.d>=\delta(s,s)$
  - $v.d=\infty\ for\ all\ v\ \in V-s$
  - $v.d>=\delta(s,v)$

- We are modifying distance of a node when it is explored for the first time
- Assuming node v is being explored through node u and for node u, $u.d >= \delta(s,u)$, we have to show it is true for v as well
- $v.d=u.d+1>= \delta(s,u)+1>= \delta(s,v)$

Lemma 3

Suppose that during the execution of BFS on a graph $G = (V, E)$, the queue Q contains the vertices $\langle v_1, v_2, \ldots, v_r \rangle$, where v_1 is the head of Q and v_r is the tail. Then, $v_r.d \leq v_1.d + 1$ and $v_i.d \leq v_{i+1}.d$ for $i = 1, 2, \ldots, r - 1$.

- After initialization, there is only s in Q, so there is no violation
- Assuming at certain stage there are $v_1, v_2, \ldots, v_r$ are there in Q and it satisfies the constraints. We have to show dequeue and enqueue does not violate this condition.
- Vr.d<=v1.d+1<=v2.d+1
- Let's say v_{r+1} node is going to be enqueued through node u.
- u.d <=v_1.d
- V_{r+1}.d=u.d+1<=v_1.d+1

Lemma 4

Suppose that vertices v_i and v_j are enqueued during the execution of BFS, and that v_i is enqueued before v_j. Then $v_i.d \leq v_j.d$ at the time that v_j is enqueued.

Lemma 5

Let $G = (V, E)$ be a directed or undirected graph, and suppose that BFS is run on G from a given source vertex $s \in V$. Then, during its execution, BFS discovers every vertex $v \in V$ that is reachable from the source s, and upon termination, $v.d = \delta(s, v)$ for all $v \in V$. Moreover, for any vertex $v \neq s$ that is reachable from s, one of the shortest paths from s to v is a shortest path from s to $v.\pi$ followed by the edge $(v.\pi, v)$.

- Assume v is assigned a distance which is not equal to its shortest path distance and v is such node with shortest $\delta(s,v)$.
- As we have already shown $v.d >= \delta(s,v)$, it implies that $v.d > \delta(s,v)$
- Let u is the preceding node in shortest path from s to v. However, $u.d = \delta(s,u)$
- $v.d > \delta(s,v) = \delta(s,u)+1 = u.d+1$

- v is white:
  - V.d=u.d+1
- v is black:
  - v is already dequeued and hence as per lemma 4, v.d<=u.d
- v is grey:
  - Assume v is colored grey when node w was dequeued. w.d=v.d-1
  - w.d<=u.d
  - v.d-1<=u.d
  - V.d<=u.d+1

Breadth-first Tree

- For a graph $G = (V, E)$ with source s, the **predecessor subgraph** of G is $G_\pi = (V_\pi, E_\pi)$ where
  - $V_\pi = \{v \in V : \pi[v] \neq \text{NIL}\} \cup \{s\}$
  - $E_\pi = \{(\pi[v], v) \in E : v \in V_\pi - \{s\}\}$
- The predecessor subgraph G_π is a **breadth-first tree** :
  - V_π consists of the vertices reachable from s and
  - for all $v \in V_\pi$, there is a unique simple path from s to v in G_π that is also a shortest path from s to v in G.
- The edges in E_π are called **tree edges**.
  $|E_\pi| = |V_\pi| - 1$.

Depth-first Search (DFS)

- Explore edges out of the most recently discovered vertex *v*.
- When all edges of *v* have been explored, backtrack to explore other edges leaving the vertex from which *v* was discovered (its *predecessor*).
- "Search as deep as possible first."
- Continue until all vertices reachable from the original source are discovered.
- If any undiscovered vertices remain, then one of them is chosen as a new source and search is repeated from that source.

Depth-first Search

- **Input:** $G = (V, E)$, directed or undirected. No source vertex given!
- **Output:**
  - 2 **timestamps** on each vertex. Integers between 1 and 2|V|.
    - $d[v]$ = ***discovery time*** (v turns from white to gray)
    - $f[v]$ = ***finishing time*** (v turns from gray to black)
  - $\pi[v]$: predecessor of $v = u$, such that v was discovered during the scan of u's adjacency list.
- Uses the same coloring scheme for vertices as BFS.

Pseudo-code

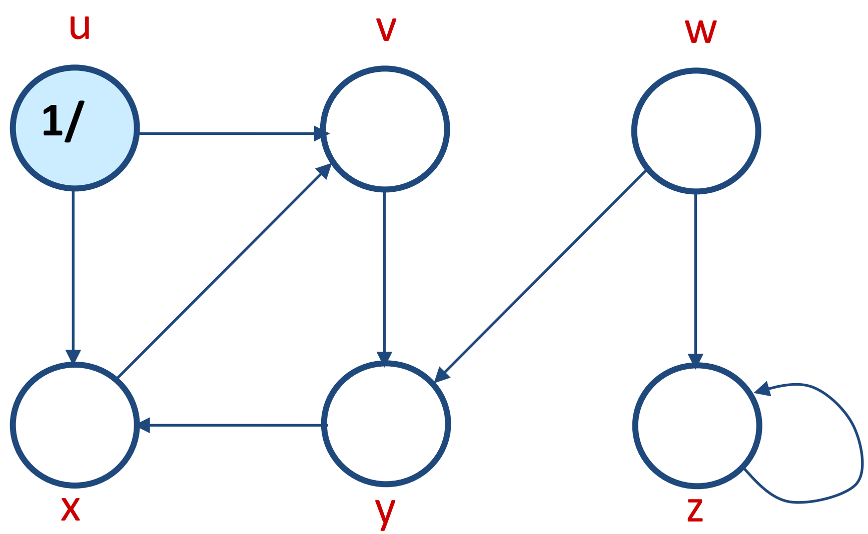
```
DFS(G)
for each vertex u ∈ V[G]
    do color[u] ← white
        π[u] ← NIL
time ← 0
for each vertex u ∈ V[G]
    do if color[u] = white
        then DFS-Visit(u)
```

```
DFS-Visit(u)
color[u] ← GRAY  ∇ White vertex u has been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]
    do if color[v] = WHITE
        then π[v] ← u
            DFS-Visit(v)
color[u] ← BLACK   ∇ Blacken u; it is finished.
f[u] ← time ← time + 1
```

Uses a global timestamp ***time***.

Example: animation.

Example (DFS)



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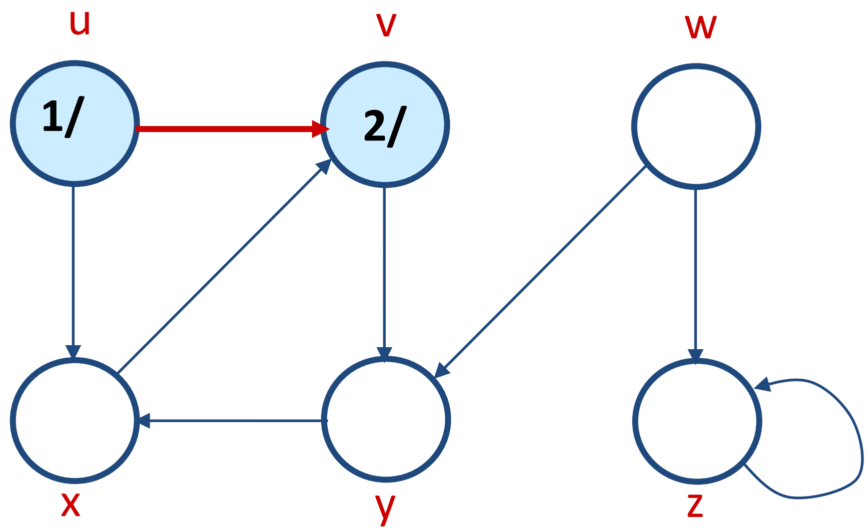

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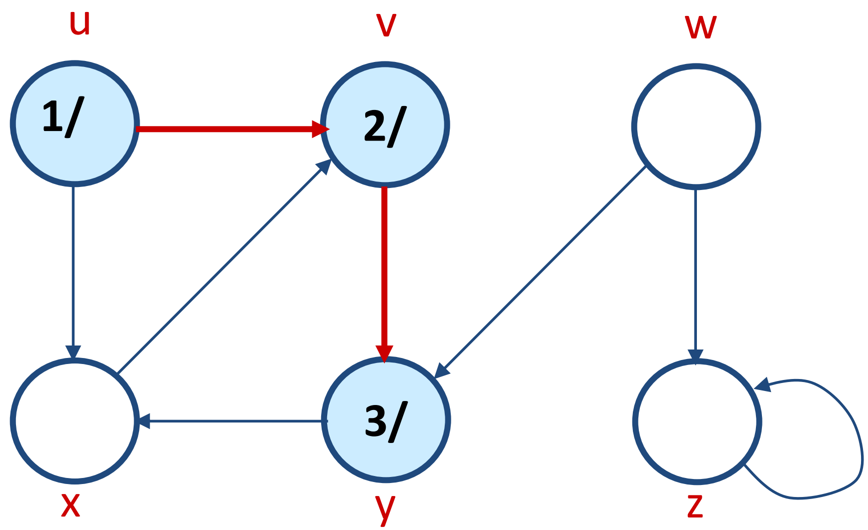

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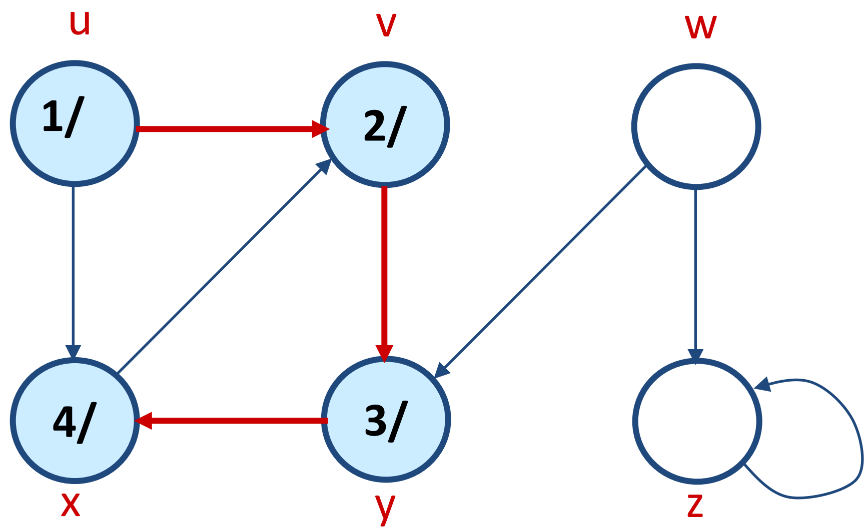

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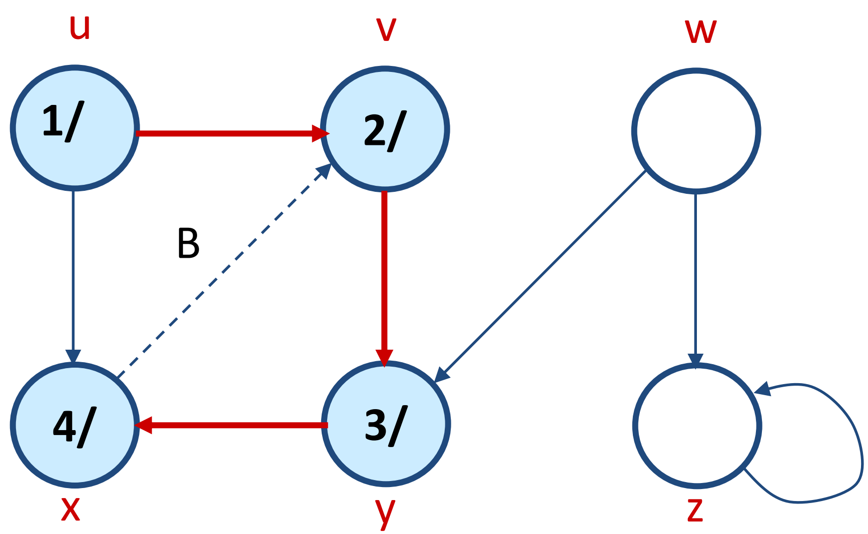

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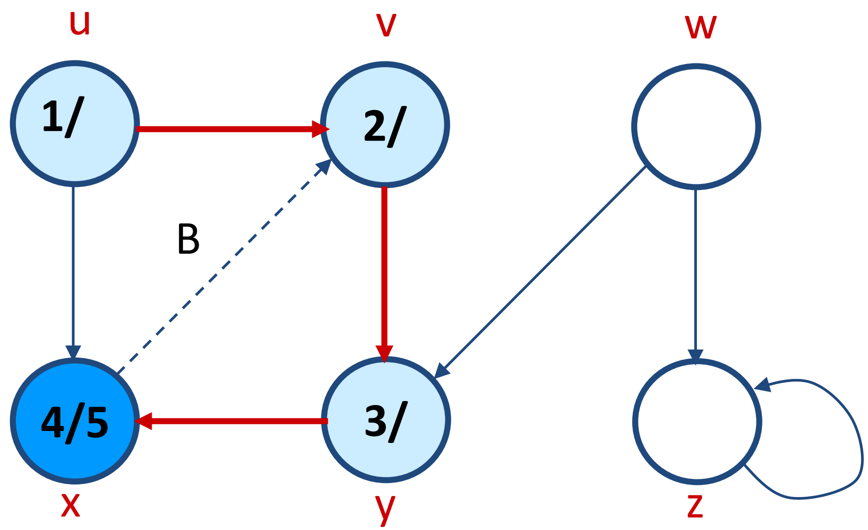

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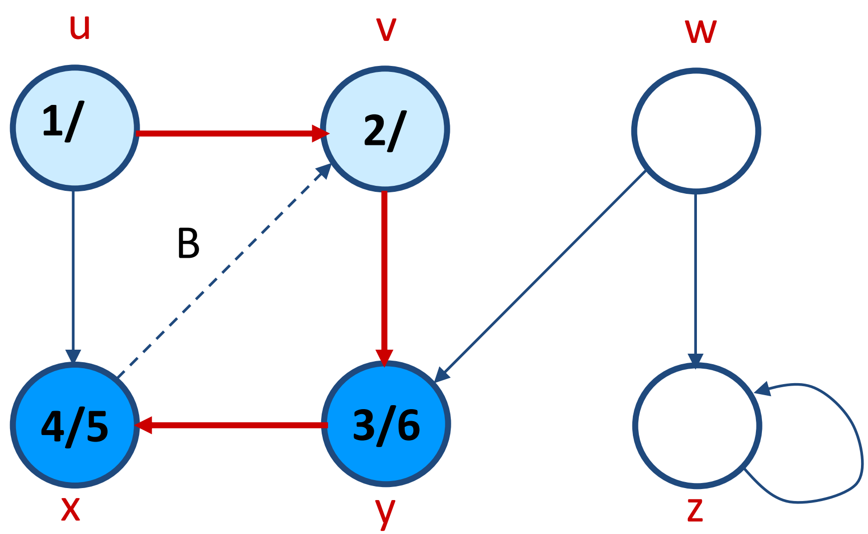

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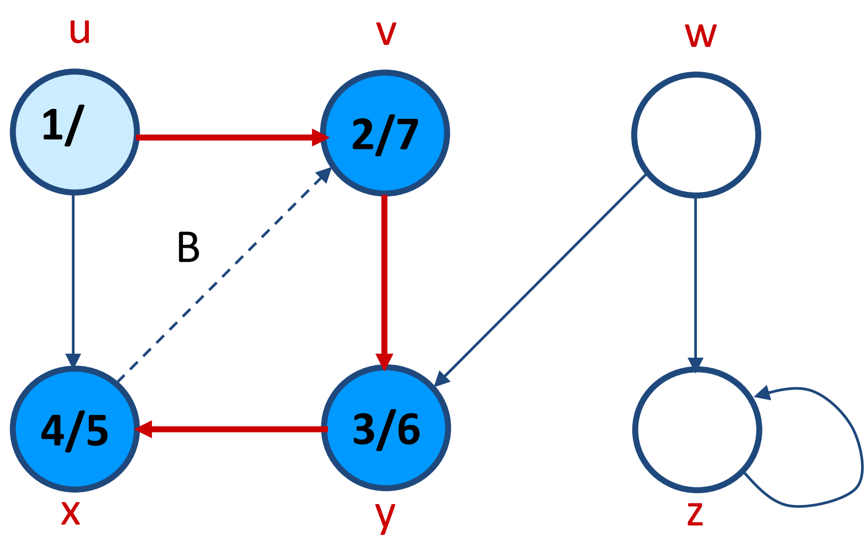

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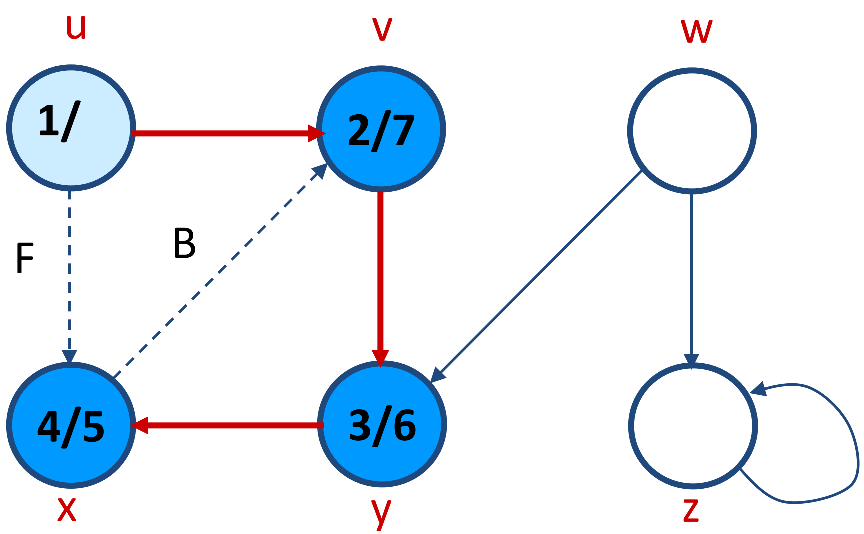

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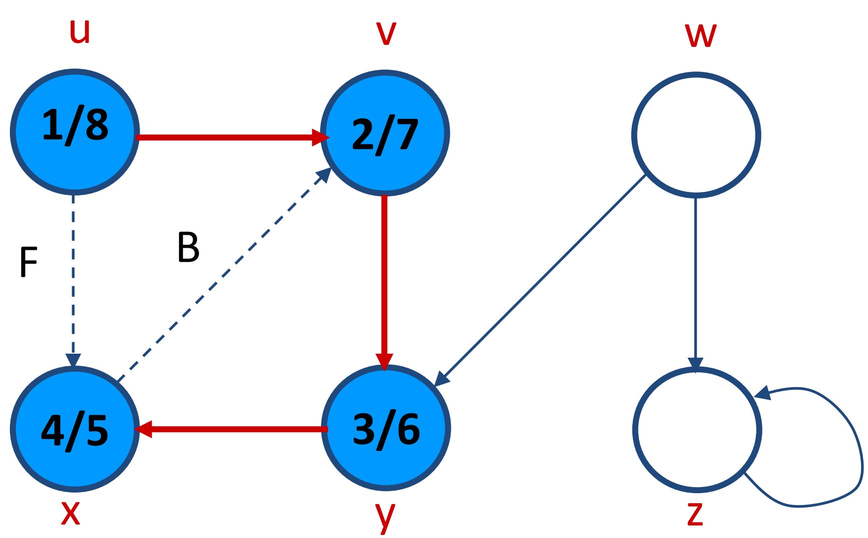

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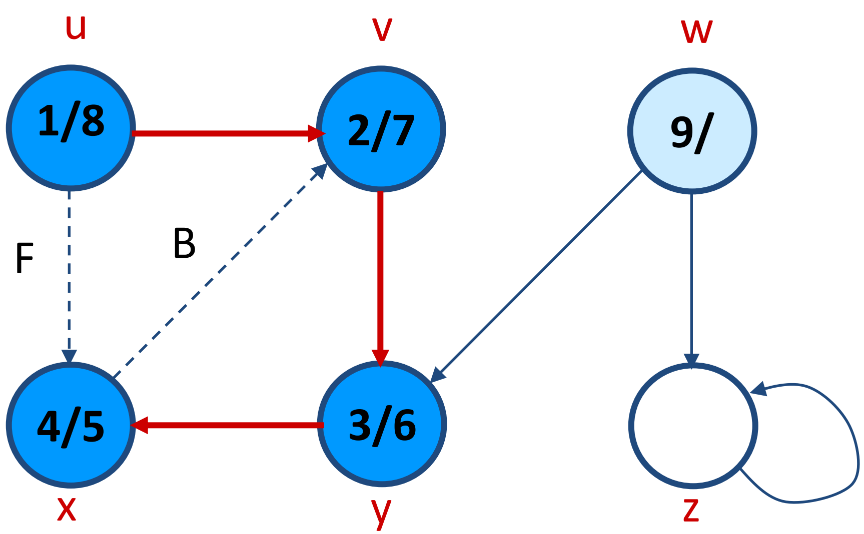

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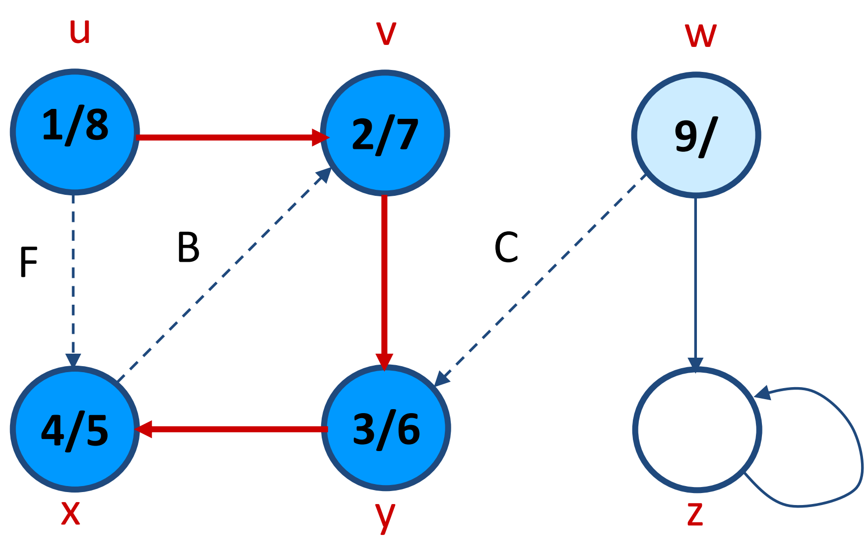

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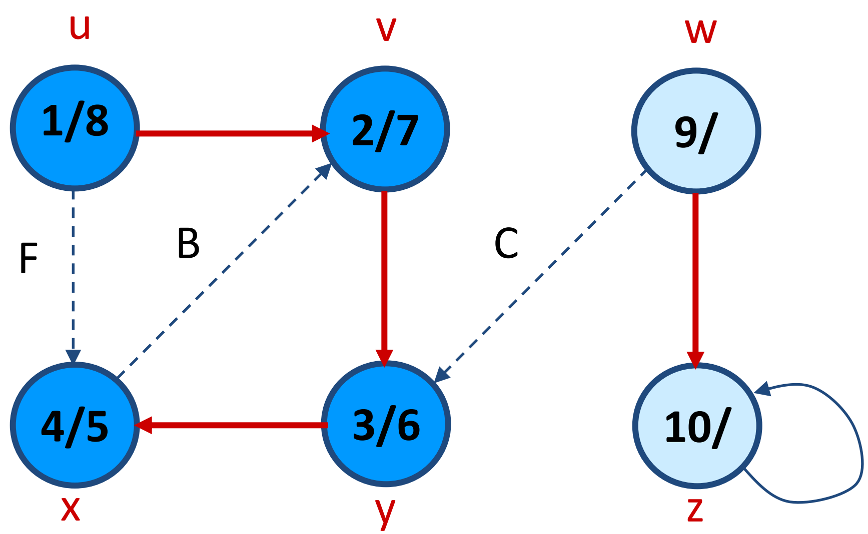

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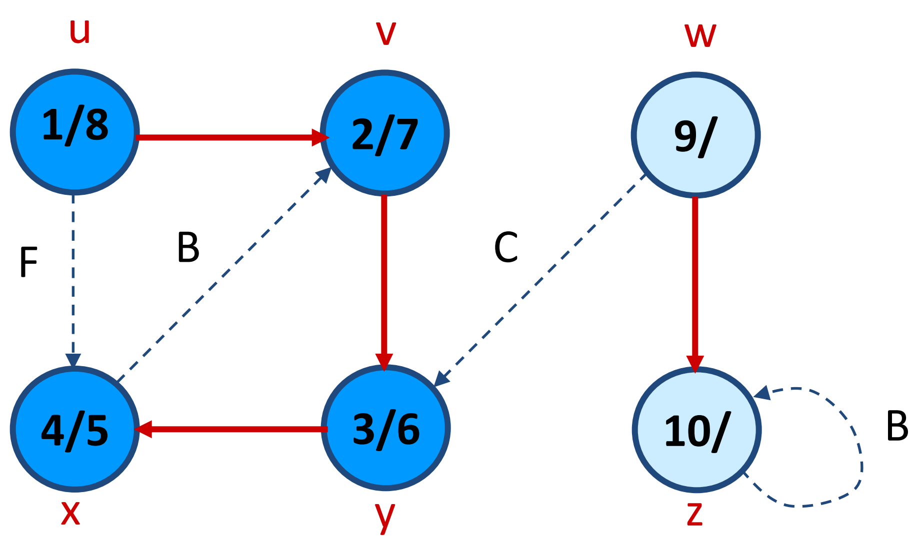

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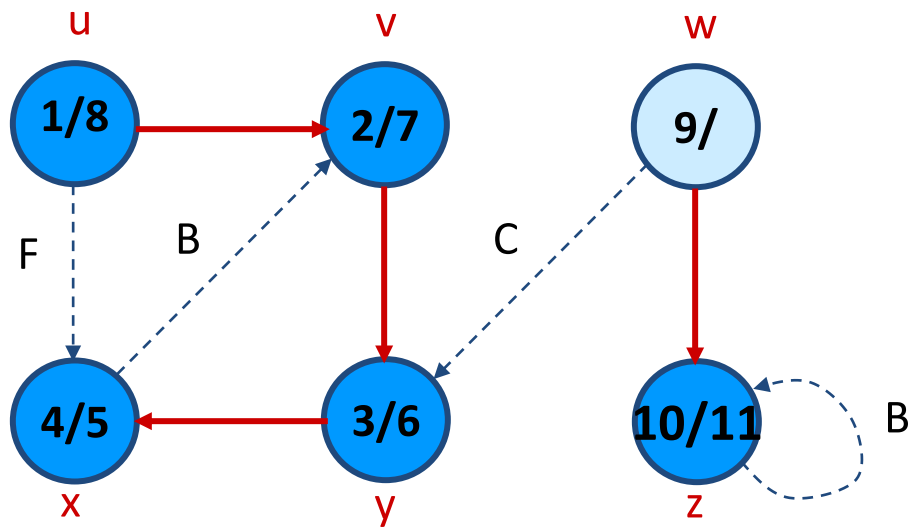

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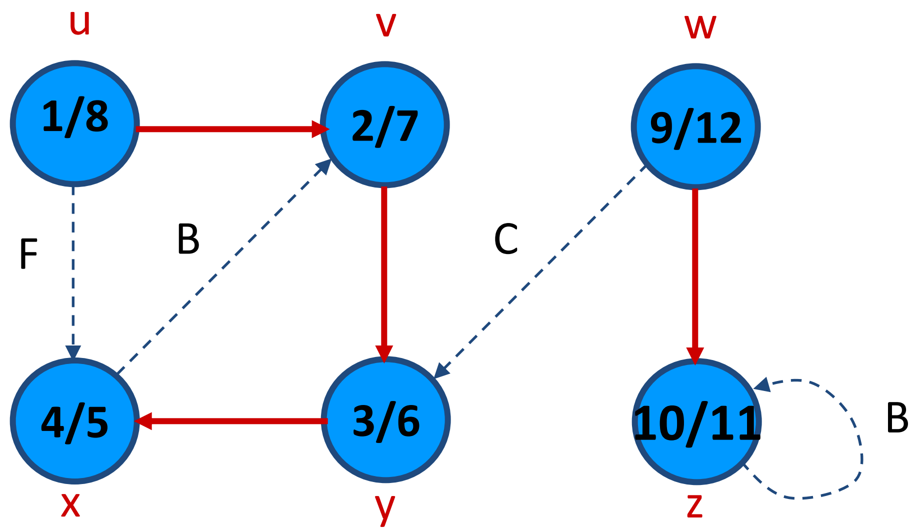


Analysis of DFS

- Loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in V$ when it's painted gray the first time. Lines 4-7 of DFS-Visit is executed $|\text{Adj}[v]|$ times. The total cost of executing DFS-Visit is $\sum_{v \in V} |\text{Adj}[v]| = \Theta(E)$
- Total running time of DFS is $\Theta(V+E)$.

<u>DFS(*G*)</u>

```
1.  for each vertex u ∈ V[G]
2.      do color[u] ← white
3.         π[u] ← NIL
4.  time ← 0
5.  for each vertex u ∈ V[G]
6.      do if color[u] = white
7.           then DFS-Visit(u)
```

DFS-Visit(u)

```
1. color[u] ← GRAY   ∇ White vertex
         u has been discovered
2. time ← time + 1
3. d[u] ← time
4. for each v ∈ Adj[u]
5.     do if color[v] = WHITE
6.          then π[v] ← u
7.               DFS-Visit(v)
8. color[u] ← BLACK     ∇ Blacken u;
         it is finished.
9.  f[u] ← time ← time + 1
```

Parenthesis Theorem

Theorem 22.7

For all u, v, exactly one of the following holds:

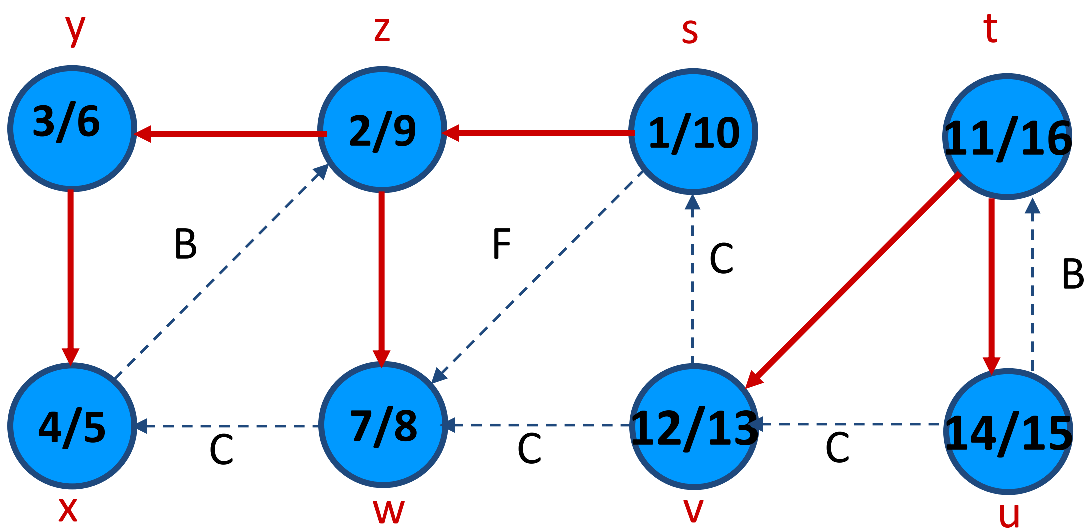
1. $d[u] < f[u] < d[v] < f[v]$ or $d[v] < f[v] < d[u] < f[u]$ and neither u nor v is a descendant of the other. => ()[] or [] ()
2. $d[u] < d[v] < f[v] < f[u]$ and v is a descendant of u.=>([])
3. $d[v] < d[u] < f[u] < f[v]$ and u is a descendant of v.=>[()]

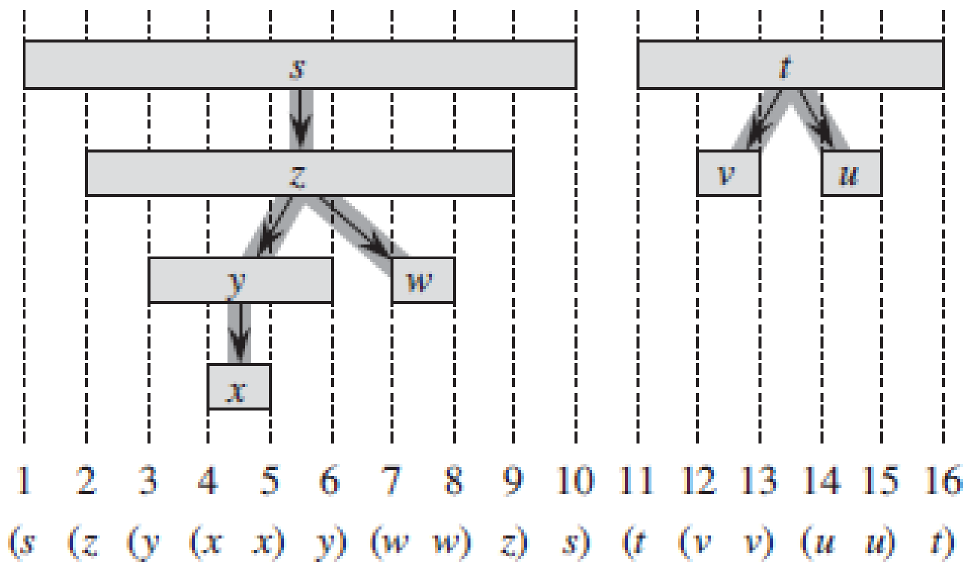
- So $d[u] < d[v] < f[u] < f[v]$ *cannot* happen.
- Like parentheses:
  - OK: ( ) [ ] ([ ]) [( )]
  - Not OK: ([ )] [( ])

Corollary

v is a proper descendant of u if and only if $d[u] < d[v] < f[v] < f[u]$.

Example (Parenthesis Theorem)





Depth-First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq \text{NIL}\}$.
  - How does it differ from that of BFS?
  - The predecessor subgraph G_π forms a *depth-first forest* composed of several *depth-first trees*. The edges in E_π are called *tree edges*.

Definition:

Forest: An acyclic graph G that may be disconnected.

White-path Theorem

Theorem 22.9

v is a descendant of u if and only if at time $d[u]$, there is a path $u \rightsquigarrow v$ consisting of only white vertices. (Except for u, which was *just* colored gray.)

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

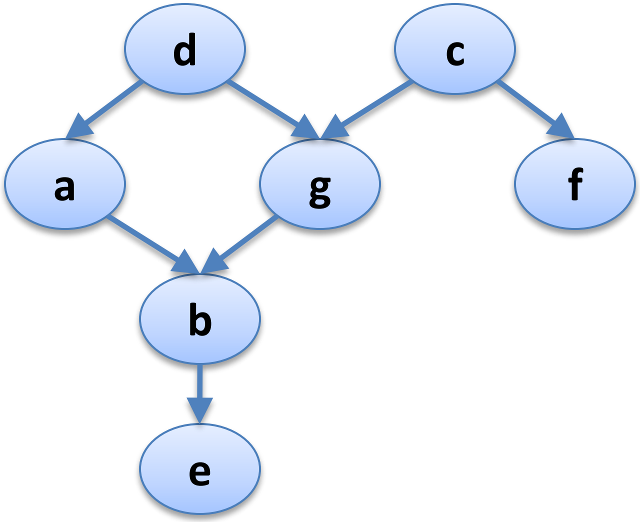
Theorem:
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Topological sort

- We have a **set of tasks** and a **set of dependencies (precedence constraints)** of form “task A must be done before task B”
- **Topological sort**: An ordering of the tasks that conforms with the given dependencies
- **Goal**: Find a topological sort of the tasks or decide that there is no such ordering

Examples

- **Scheduling**: When scheduling *task graphs* in distributed systems, usually we first need to sort the tasks topologically
  ...and then assign them to resources
- Or during compilation to order modules/libraries



Examples

- **Resolving dependencies**: *apt-get* uses topological sorting to obtain the admissible sequence in which a set of Debian packages can be installed/removed

Topological sort more formally

- Suppose that in a **directed** graph **G = (V, E)** vertices **V** represent tasks, and each edge (**u**, **v**)∈**E** means that task **u** must be done before task **v**

- What is an ordering of vertices 1, ..., |**V**| such that for every edge (**u**, **v**), **u** appears before **v** in the ordering?

- Such an ordering is called a **topological sort of G**
- Note: there can be multiple topological sorts of G

Topological sort more formally

- Is it possible to execute all the tasks in **G** in an order that respects all the precedence requirements given by the graph edges?
- The answer is "**yes**" *if and only if* the directed graph **G** has **no cycle**!

  (otherwise we have a **deadlock**)
- Such a **G** is called a Directed Acyclic Graph, or just a **DAG**

Algorithm for TS

- TOPOLOGICAL-SORT(**G**):
  1) call DFS(G) to compute **finishing** times **f**[**v**] for each vertex **v**
  2) as each vertex is finished, insert it onto the **front** of a linked list
  3) return the linked list of vertices

- Note that the result is just a list of vertices in order of **decreasing** finish times **f**[]

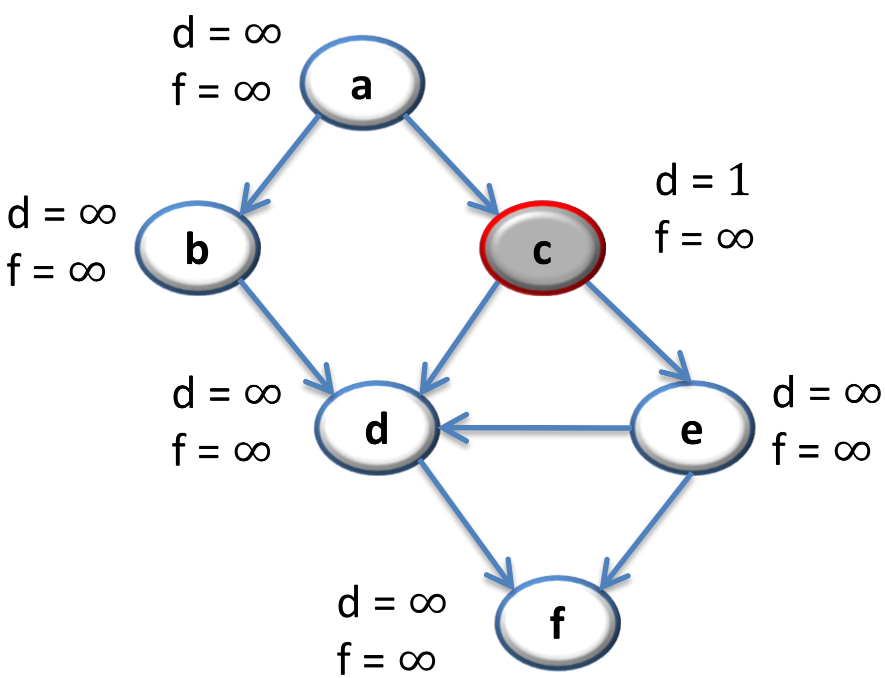
DAGs and back edges

- Can there be a **back** edge in a DFS on a DAG?
- NO! Back edges close a cycle!
- A graph **G** is a DAG <=> there is no back edge classified by DFS(**G**)

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 1

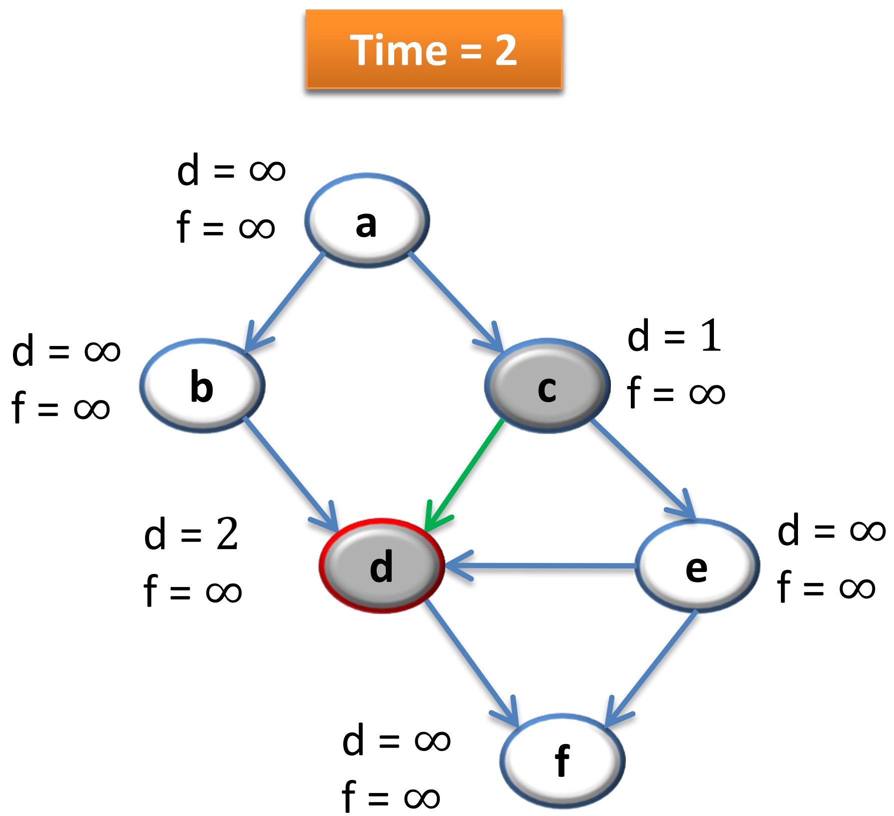


Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

Topological sort

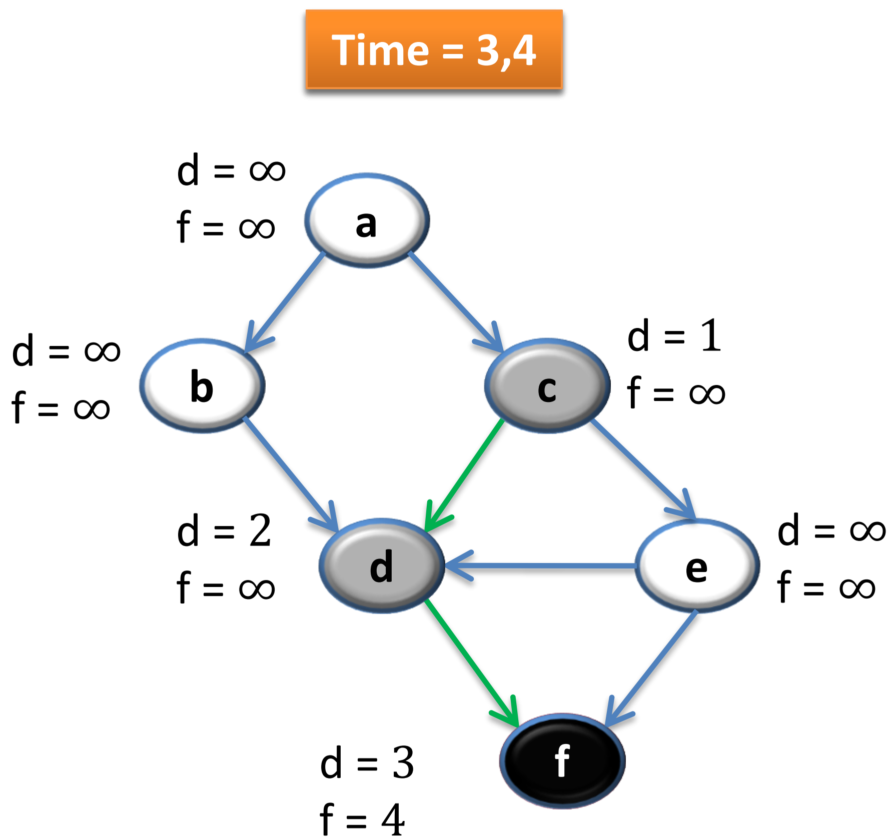
1) Call DFS(**G**) to compute the finishing times **f**[**v**]



Let’s say we start the DFS from the vertex **c**

Next we discover the vertex **d**

Topological sort

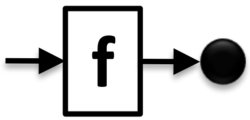


1) Call DFS(**G**) to compute the finishing times **f**[**v**]

2) as each vertex is finished, insert it onto the **front** of a linked list

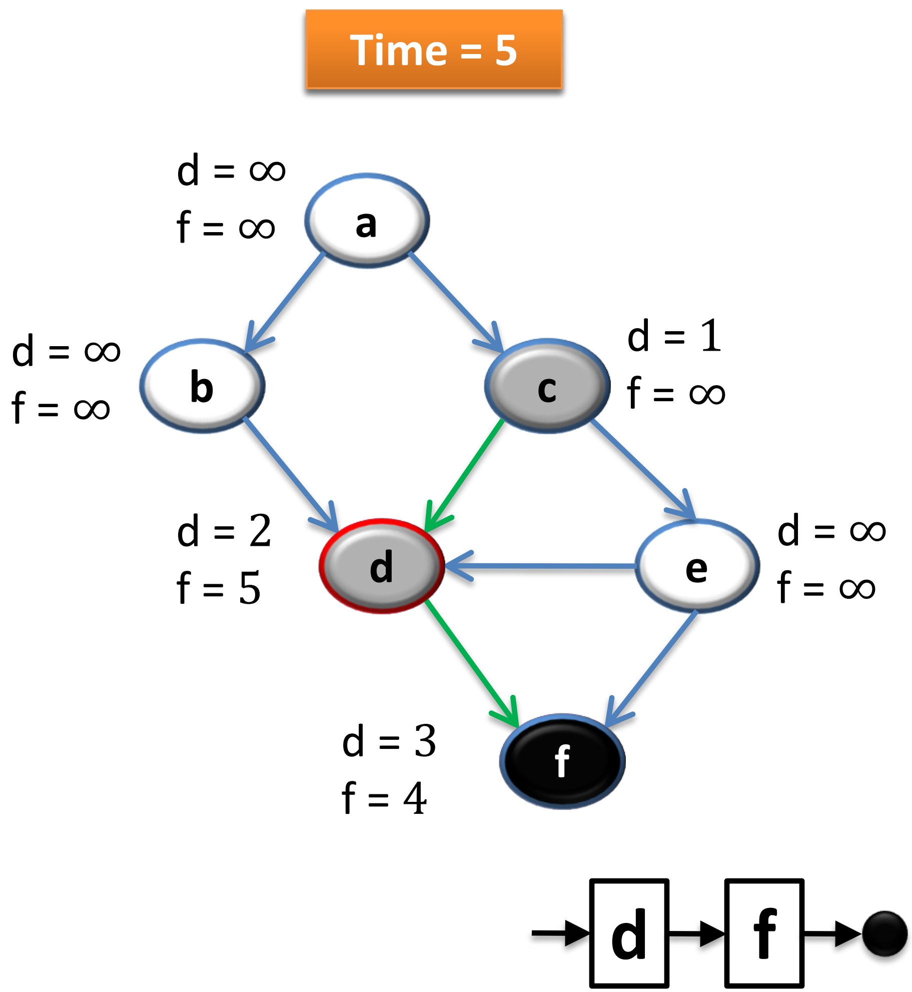
Next we discover the vertex **f**

f is done, move back to **d**



Topological sort

1) Call DFS(**G**) to compute the finishing times **f[v]**



Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

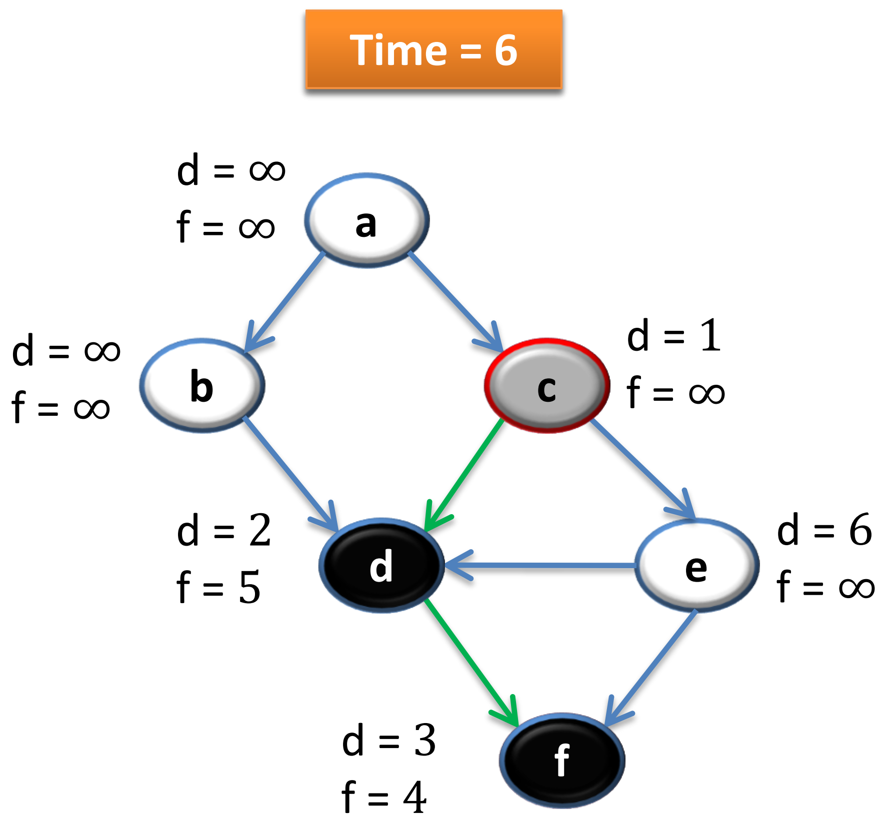
Next we discover the vertex **f**

f is done, move back to **d**

d is done, move back to **c**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]



Let's say we start the DFS from the vertex **c**

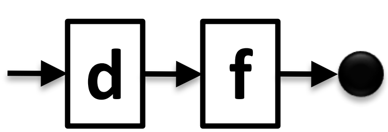
Next we discover the vertex **d**

Next we discover the vertex **f**

f is done, move back to **d**

d is done, move back to **c**

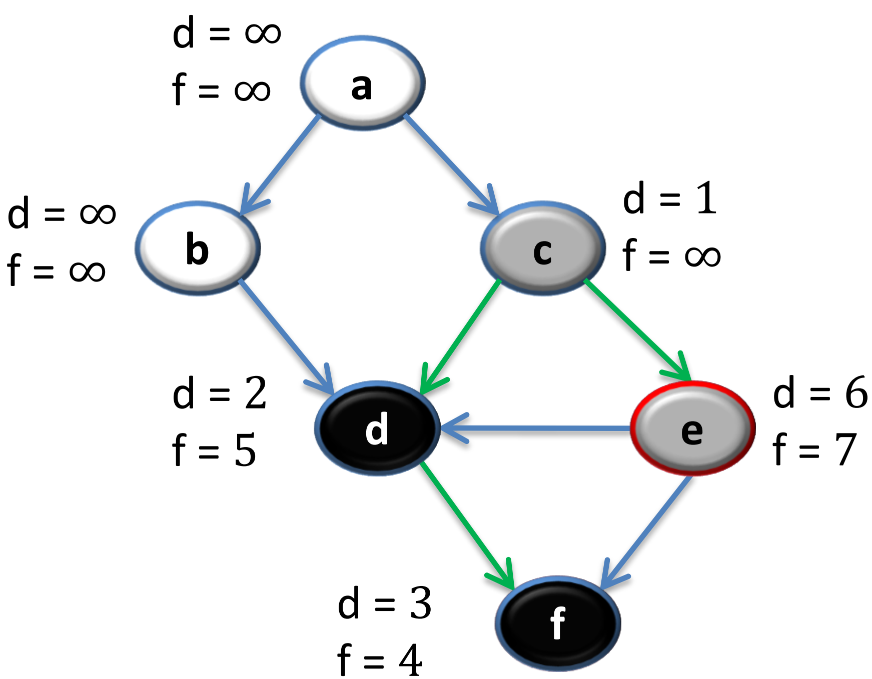
Next we discover the vertex **e**

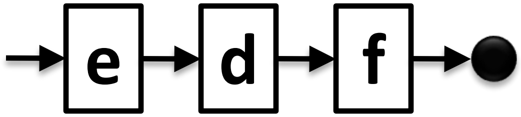


Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 7





Let's say we start the DFS from the vertex **c**

Next we discover the vertex **d**

d is done, move back to **c**

Next we discover the vertex **e**

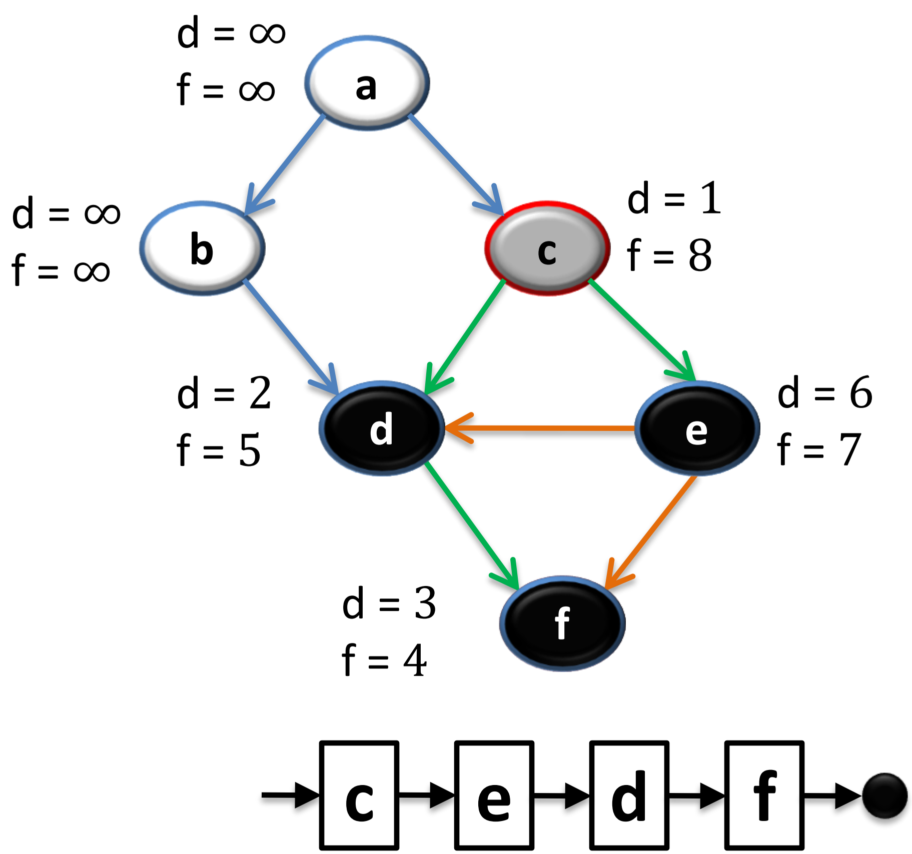
e is done, move back to **c**

Both edges from **e** are **cross edges**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

Time = 8



Let's say we start the DFS from the vertex **c**

<u>Just a note:</u> If there was (**c**,**f**) edge in the graph, it would be classified as a **forward edge** (in this particular DFS run)

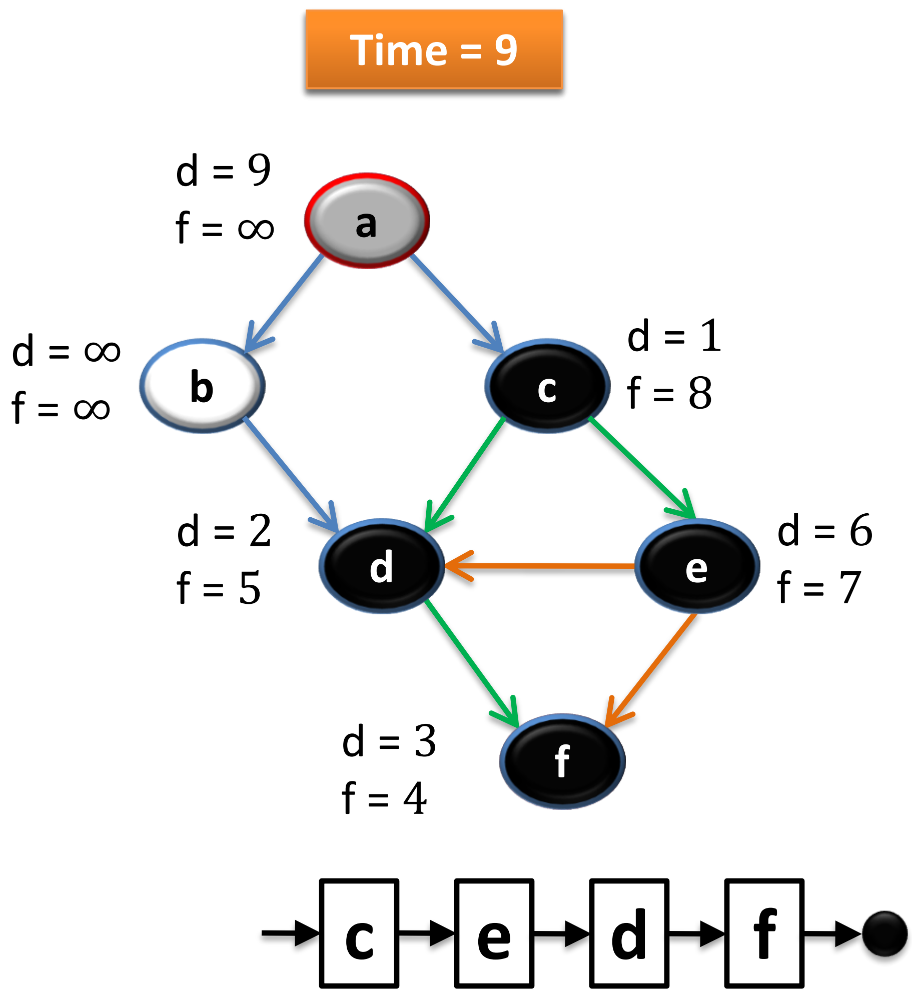
d is done, move back to **c**

Next we discover the vertex **e**

e is done, move back to **c**

c is done as well

Topological sort



1) Call DFS(**G**) to compute the finishing times **f**[**v**]

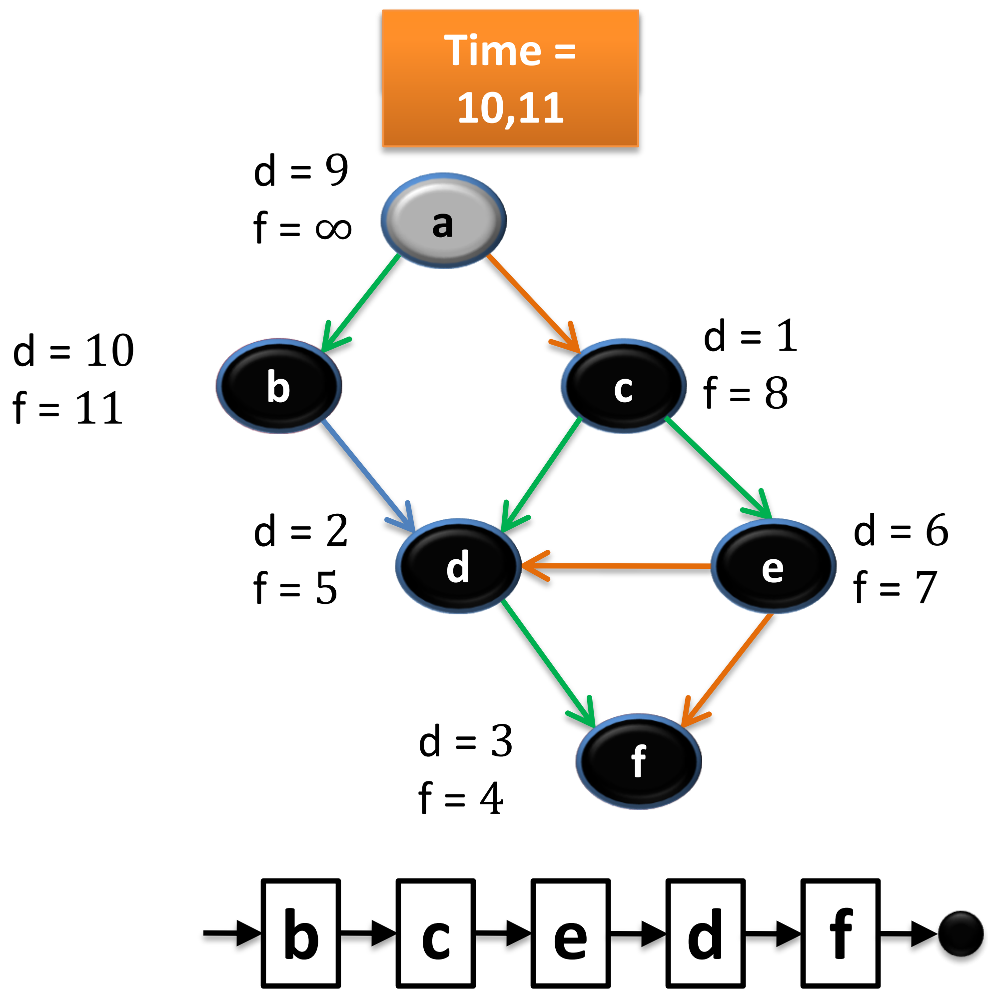
Let's now call DFS visit from the vertex **a**

Next we discover the vertex **c**, but **c** was already processed => (**a**,**c**) is a cross edge

Next we discover the vertex **b**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]



Let's now call DFS visit from the vertex **a**

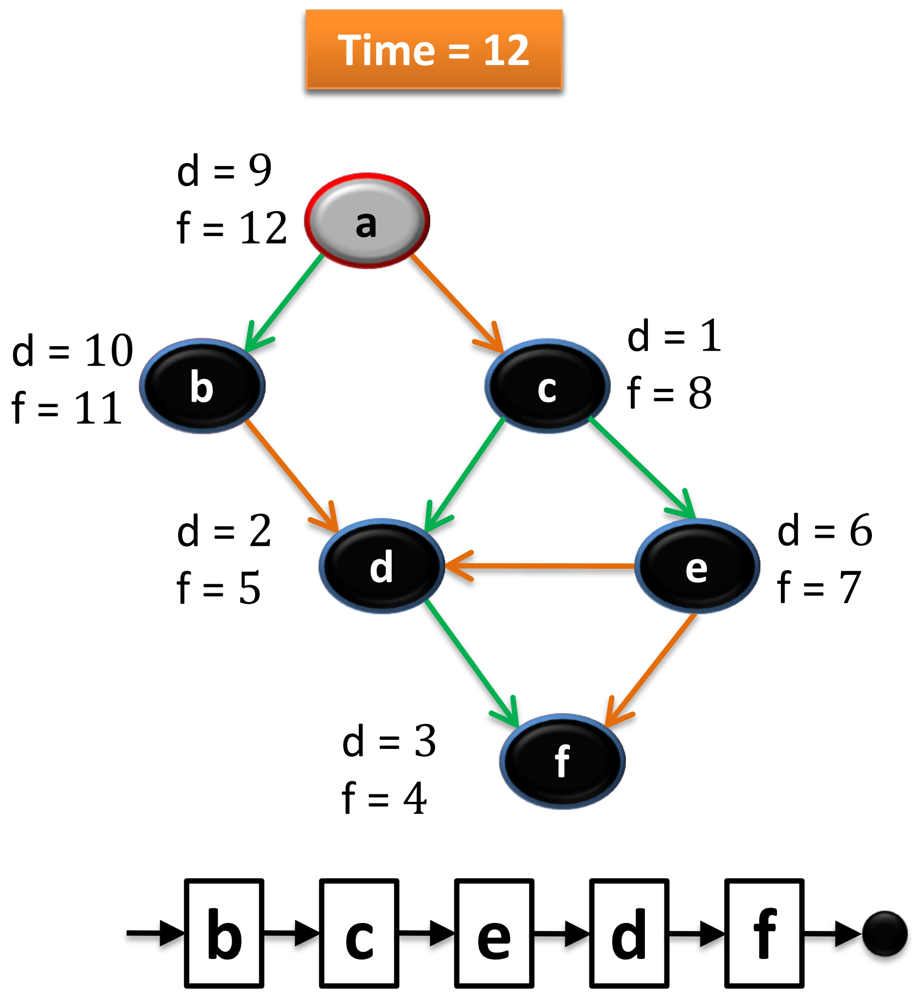
Next we discover the vertex **c**, but **c** was already processed => (**a**,**c**) is a cross edge

Next we discover the vertex **b**

b is done as (**b**,**d**) is a cross edge => now move back to **c**

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]



Let's now call DFS visit from the vertex **a**

Next we discover the vertex **c**, but **c** was already processed => (**a**,**c**) is a cross edge

Next we discover the vertex **b**

b is done as (**b**,**d**) is a cross edge => now move back to **c**

a is done as well

Topological sort

1) Call DFS(**G**) to compute the finishing times **f**[**v**]

WE HAVE THE RESULT!

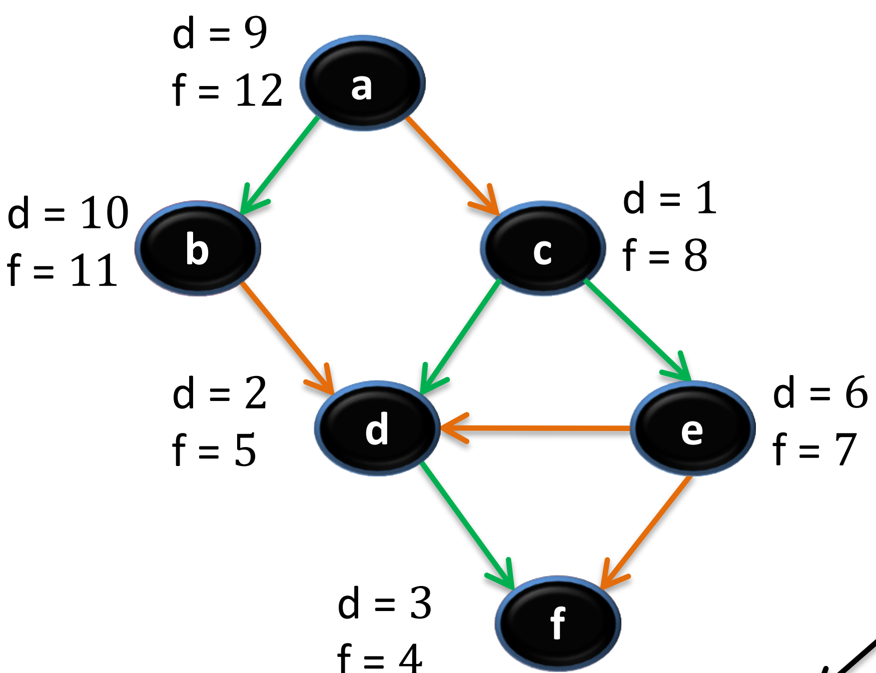
3) return the linked list of vertices

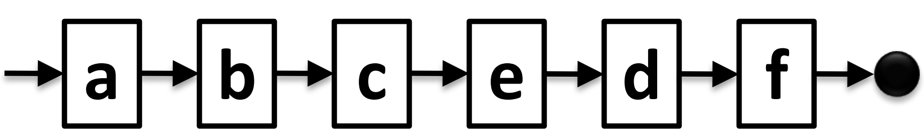
=> (**a**,**c**) is a cross edge

Next we discover the vertex **b**

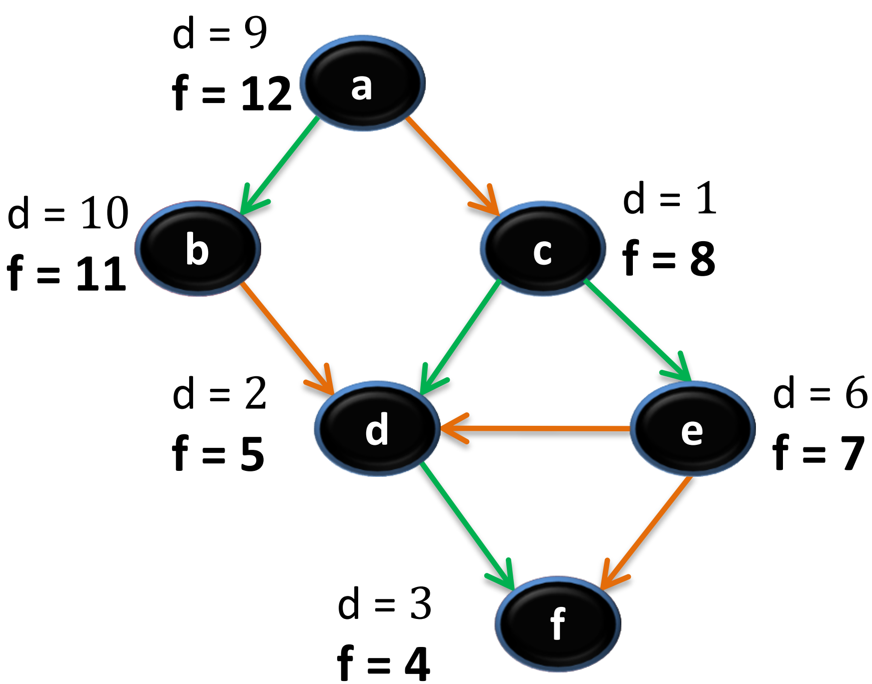
b is done as (**b**,**d**) is a cross edge => now move back to **c**

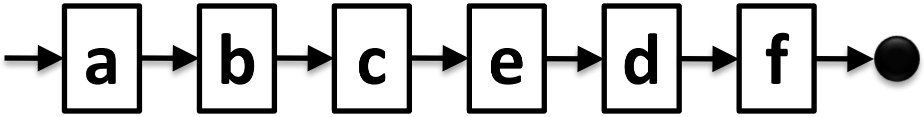
a is done as well





Topological sort





The linked list is sorted in **decreasing** order of finishing times **f**[]

Try yourself with different vertex order for DFS visit

Time complexity of TS(G)

- Running time of topological sort:

  $$\boldsymbol{\Theta(V + E)}$$

  Why? Depth first search takes $\boldsymbol{\Theta(V + E)}$ time in the worst case, and inserting into the front of a linked list takes $\boldsymbol{\Theta(1)}$ time

Proof of correctness

- **Theorem**: TOPOLOGICAL-SORT(**G**) produces a topological sort of a DAG **G**

- The TOPOLOGICAL-SORT(**G**) algorithm does a DFS on the DAG **G**, and it lists the nodes of **G** in order of decreasing finish times **f**[]
- We must show that this list satisfies the topological sort property, namely, that for every edge (**u**,**v**) of **G**, **u** appears before **v** in the list
- **Claim**: For every edge (**u**,**v**) of **G**: **f**[**v**] < **f**[**u**] in DFS

Proof of correctness

“For every edge (**u**,**v**) of **G**, **f**[**v**] < **f**[**u**] in this DFS”

- The DFS classifies (**u**,**v**) as a **tree edge**, a **forward edge** or a **cross-edge** (it cannot be a back-edge since **G** has no cycles):
  1. If (**u**,**v**) is a **tree** or a **forward edge** $\Rightarrow$ **v** is a descendant of **u** $\Rightarrow$ **f**[**v**] < **f**[**u**]
  2. If (**u**,**v**) is a **cross-edge =>** v will be discovered and finished before u hence f[v] < f[u]

Proof of correctness

- TOPOLOGICAL-SORT(G) lists the nodes of G from highest to lowest finishing times

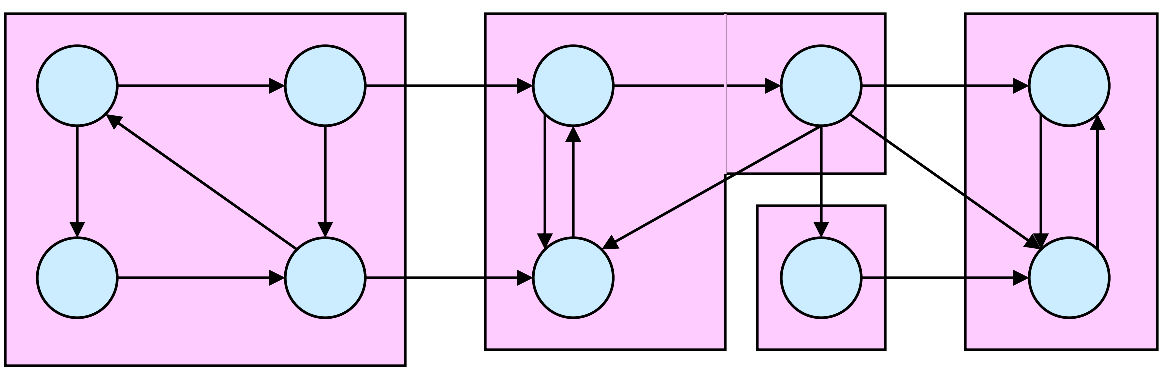
- By the **Claim**, for every edge (**u**,**v**) of **G**:

$$\mathbf{f}[\mathbf{v}] < \mathbf{f}[\mathbf{u}]$$

$\Rightarrow$ **u** will appear before **v** in the sorted list

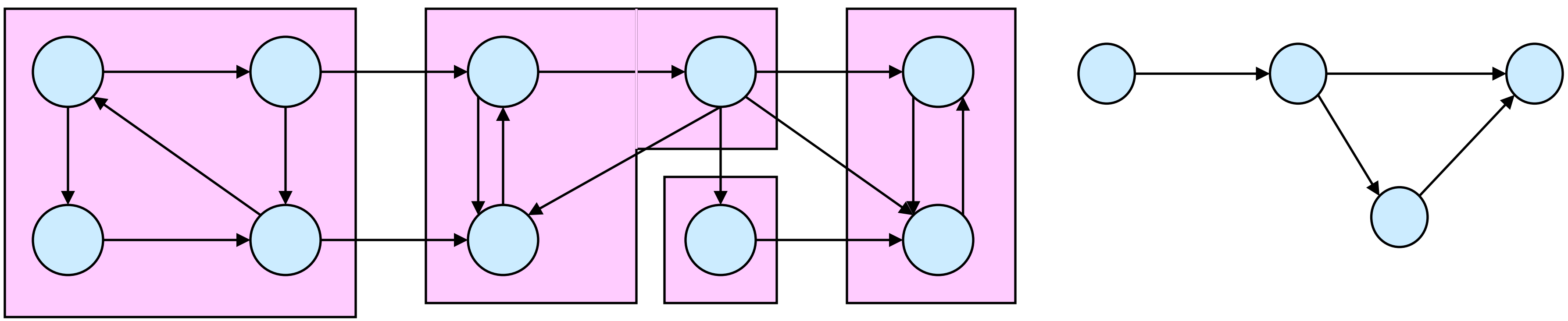
Strongly Connected Components

- G is strongly connected if every pair (u, v) of vertices in G is reachable from one another.
- A **strongly connected component** (***SCC***) of G is a maximal set of vertices $C \subseteq V$ such that for all u, $v \in C$, both $u \leadsto v$ and $v \leadsto u$ exist.



Component Graph

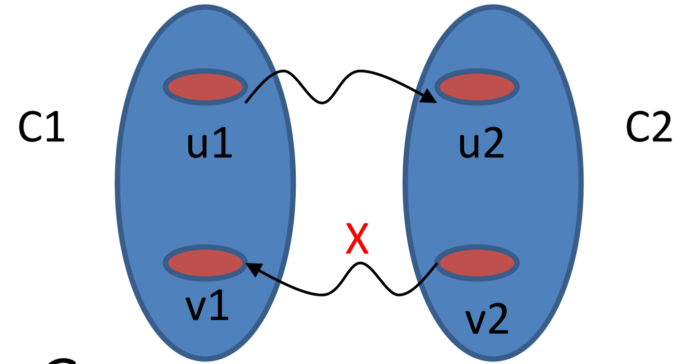
- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- V^{SCC} has one vertex for each SCC in G.
- E^{SCC} has an edge if there's an edge between the corresponding SCC's in G.
- G^{SCC} for the example considered:



G^{SCC} is a DAG

Lemma

Let C and C' be distinct SCC's in G, let $u1$, $v1 \in C1$ and $u2$, $v2 \in C2$, and suppose there is a path $u1 \rightsquigarrow u2$ in G. Then there cannot also be a path $v2 \rightsquigarrow v1$ in G.



Proof:

- Suppose there is a path $v2 \rightsquigarrow v1$ in G.
- Then there are paths $v1 \rightsquigarrow u1 \rightsquigarrow u2 \rightsquigarrow v2$ in G.
- Therefore, $v1$ and $v2$ are reachable from each other, so they are not in separate SCC's.

Transpose of a Directed Graph

- G^{T} = **transpose** of directed G.
  - $G^{\mathrm{T}} = (V, E^{\mathrm{T}}), E^{\mathrm{T}} = \{(u, v) : (v, u) \in E\}$.
  - G^{T} is G with all edges reversed.
- Can create G^{T} in $\Theta(V + E)$ time if using adjacency lists.
- G and G^{T} have the *same* SCC's. (u and v are reachable from each other in G if and only if reachable from each other in G^{T}.)

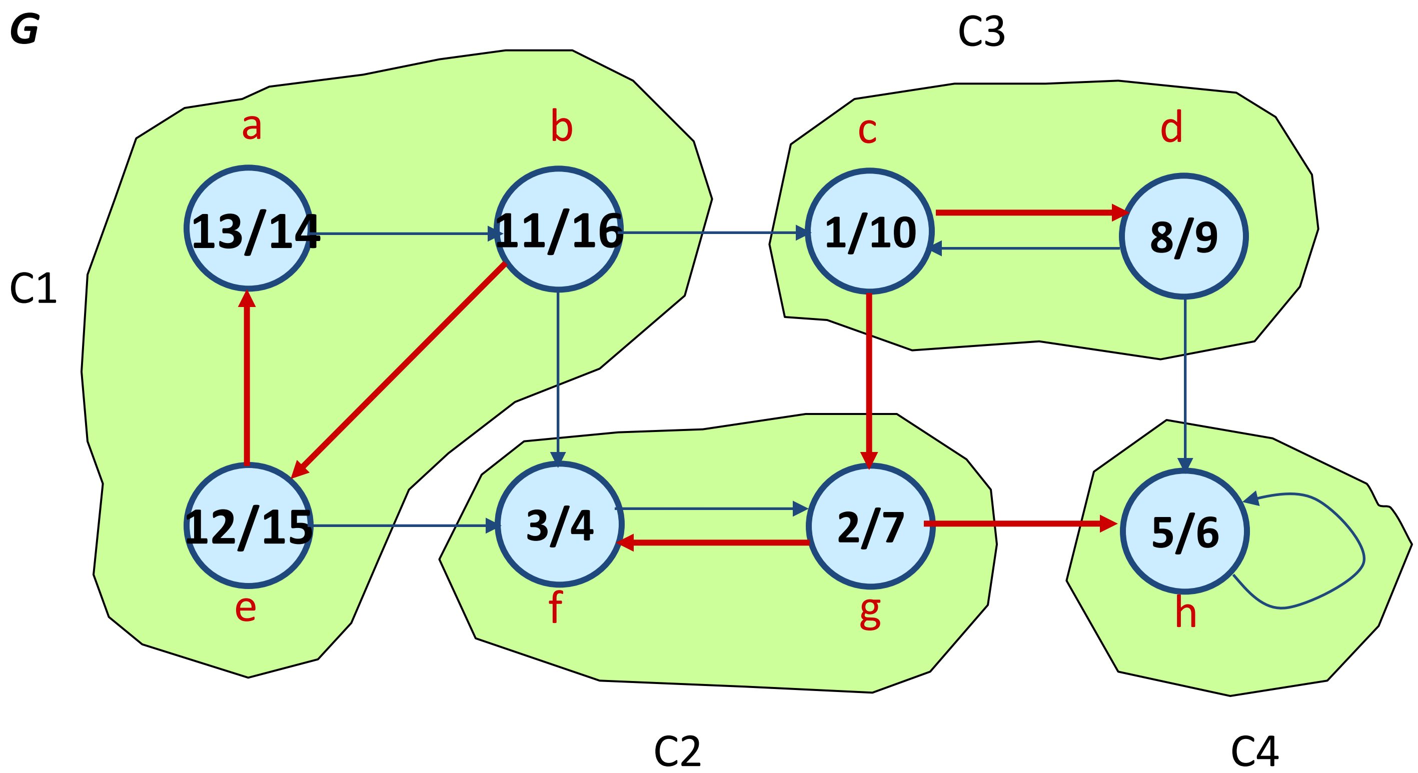
Algorithm to determine SCCs

<u>SCC(G)</u>

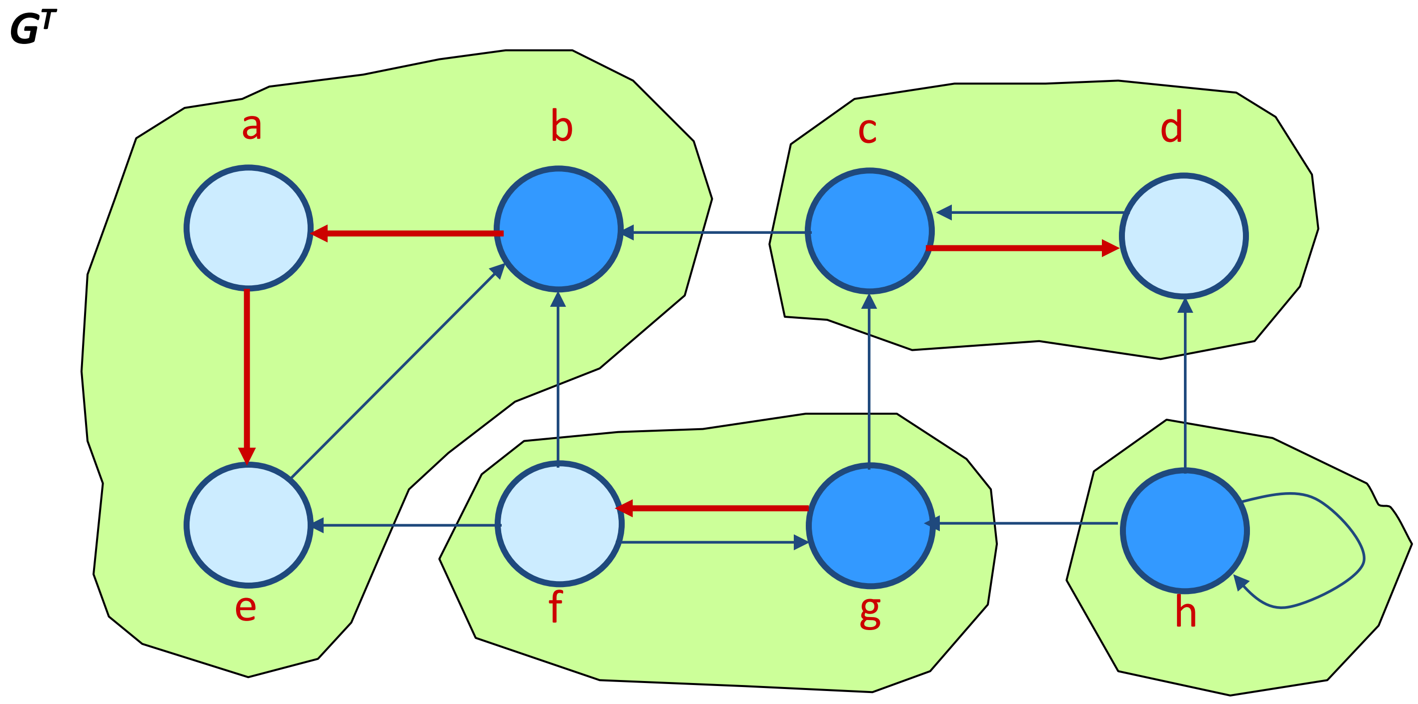
1. call DFS(G) to compute finishing times $f[u]$ for all u
2. compute G^{T}
3. call DFS(G^{T}), but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in first DFS)
4. output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

Time: $\Theta(V + E)$.

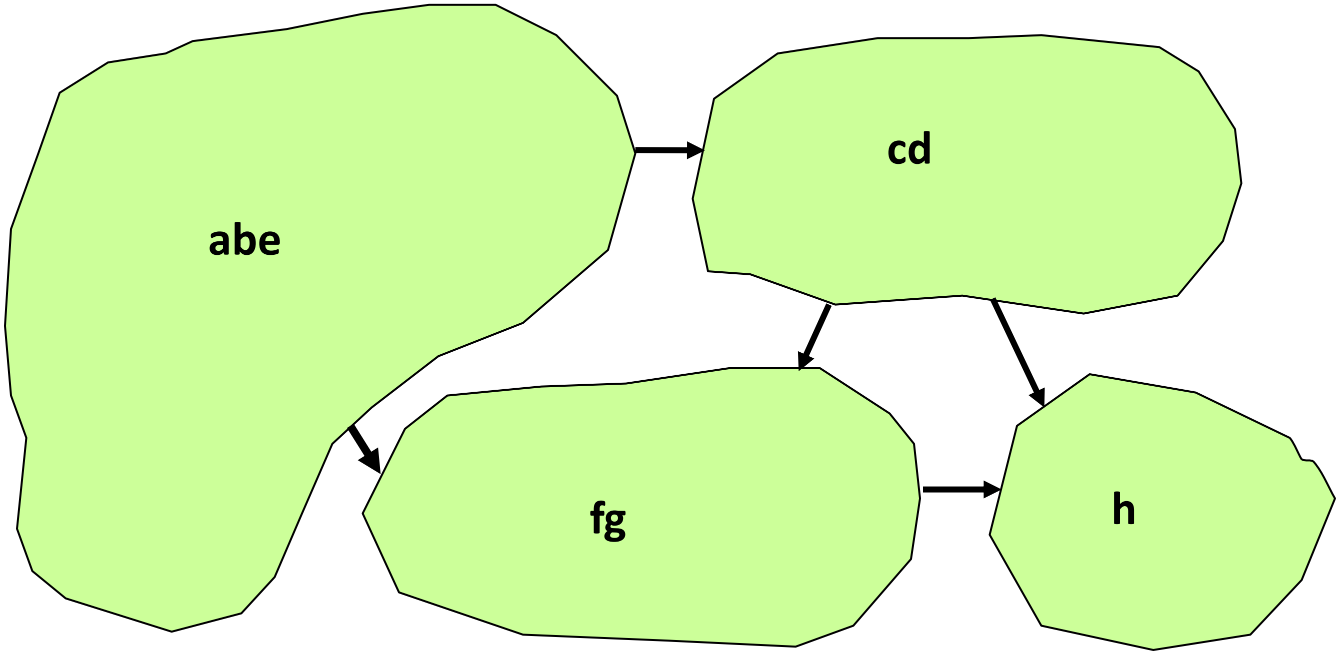
Example



Example



Example



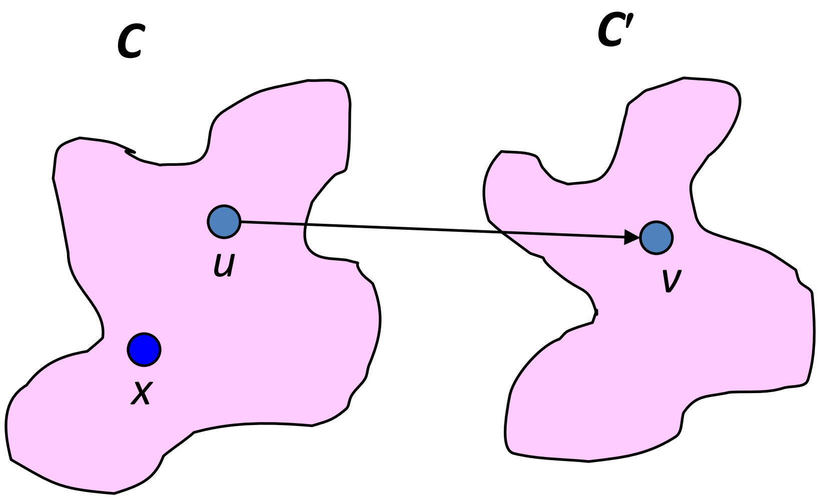
SCCs and DFS finishing times

Lemma 22.14

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

- Case 1: $d(C) < d(C')$
  - Let x be the first vertex discovered in C.
  - At time $d[x]$, all vertices in C and C' are white. Thus, there exist paths of white vertices from x to all vertices in C and C'.
  - By the white-path theorem, all vertices in C and C' are descendants of x in depth-first tree.
  - By the parenthesis theorem, $f[x] = f(C) > f(C')$.



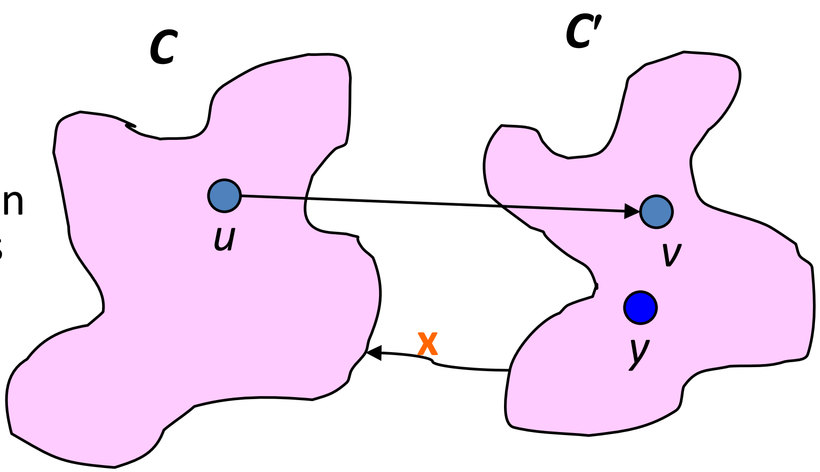
SCCs and DFS finishing times

Lemma 22.14

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

- Case 2: $d(C) > d(C')$
  - Let y be the first vertex discovered in C'.
  - At time $d[y]$, all vertices in C' are white and there is a white path from y to each vertex in C' $\Rightarrow$ all vertices in C' become descendants of y. Again, $f[y] = f(C')$.
  - At time $d[y]$, all vertices in C are also white.
  - By earlier lemma, since there is an edge (u, v), we cannot have a path from C' to C.
  - So no vertex in C is reachable from y.
  - Therefore, at time $f[y]$, all vertices in C are still white.
  - Therefore, for all $w \in C$, $f[w] > f[y]$, which implies that $f(C) > f(C')$.



Correctness of SCC

- When we do the second DFS, on G^T, start with SCC C such that $f(C)$ is maximum.
  - The second DFS starts from some $x \in C$, and it visits all vertices in C.
  - Corollary 22.15 says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
  - Therefore, DFS will visit *only* vertices in C.
  - Which means that the depth-first tree rooted at x contains *exactly* the vertices of C.

Correctness of SCC

- The next root chosen in the second DFS is in SCC C' such that $f(C')$ is maximum over all SCC's other than C.
  - DFS visits all vertices in C', but the only edges out of C' go to C, *which we've already visited*.
  - Therefore, the only tree edges will be to vertices in C'.
- We can continue the process.
- Each time we choose a root for the second DFS, it can reach only
  - vertices in its SCC—get tree edges to these,
  - vertices in SCC's *already visited* in second DFS—get *no* tree edges to these.