EE549 - Power System Dynamics and Control

Small Signal Stability

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Small Signal Stability

It is the ability of the power system to maintain synchronism when subjected to small disturbances.

- Small disturbances can be a small change in load or generation or line tripping.
- Since the disturbance is small, the nonlinear differential algebraic equations that describe the system may be linearized around the steady operating point.
- Instability can be of two forms.
    1. steady increase in rotor angle due to lack of synchronizing torque
    2. rotor oscillations of increasing amplitude due to lack of sufficient damping torque.
- In modern power systems, the instability is mainly due to insufficient damping.

There are three modes of oscillations due to small disturbance.

1. Local modes of oscillations are due to a single generator or group of generators oscillating against the rest of the system.
2. Intra-plant modes of oscillations are due to oscillations among the generators in the same plant. The typical frequencies of oscillations of local and intra plant modes are in the range of 1 Hz to 2 Hz.
3. Inter area modes of oscillations are due to a group of generators in one area oscillating together against another group of generators in a different area. The typical frequency range of inter area mode of oscillations are 0.1 Hz to 0.8 Hz.

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- Local, intra-plant and inter-area are electromechanical modes of oscillations in which the rotor angle and the speed of the generators oscillate.
- Apart from electromechanical modes of oscillations there can be control modes, due to lack of proper tuning of controller like voltage regulator of an excitation system, and torsional mode of oscillations due to oscillations in the turbine-generator shafts.

Fundamental Concepts of Stability of Dynamic Systems

The behavior of a dynamic system such as a power system may be represented by a set of n first order nonlinear ODEs as follows:

$$\dot{x}_i = f_i(x_1, x_2, \cdots, x_n; u_1, u_2, \cdots, u_r; t) \quad i = 1, 2, \cdots, n \tag{1}$$

where n is the order of the system and r is the number of inputs. It can be written as

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}, \mathrm{u}, t) \tag{2}$$

where

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathrm{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \quad \mathrm{f} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

If the state variables are not explicit functions of time, the system is said to be autonomous.

$$\dot{\mathrm{x}} = \mathrm{f}(\mathrm{x}, \mathrm{u}) \tag{3}$$

The output vector can also be written as

$$\mathrm{y} = \mathrm{g}(\mathrm{x}, \mathrm{u}) \tag{4}$$

where

$$\mathrm{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad \mathrm{g} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix}$$

Equilibrium Points

At equilibrium,

$$\dot{x} = f(x_0, u_0) = 0 \qquad (5)$$

where x_0 is the state vector x at the equilibrium point.

- If f is a set of linear functions, the system is linear. Linear systems have only one equilibrium state.
- If f is a set of nonlinear functions, the system is nonlinear. Nonlinear systems may have more than one equilibrium state.

Linearization

Let x_0 be the initial state vector and u_0 be the input vector corresponding to the equilibrium point.

$$\dot{x} = f(x_0, u_0) = 0 \tag{6}$$

Let us perturb the system from the above state.

$$x = x_0 + \Delta x \quad u = u_0 + \Delta u$$

where Δ denotes a small deviation.
At the new state,

$$\begin{aligned} \dot{x} &= \dot{x_0} + \Delta\dot{x} \\ &= f(x_0 + \Delta x, u_0 + \Delta u) \end{aligned} \tag{7}$$

As the perturbations are assumed to be small, the nonlinear functions $\mathrm{f}(\mathrm{x}, \mathrm{u})$ can be expressed in terms of Taylor's series expansion. By neglecting the higher order terms,

$$\begin{aligned}\dot{x}_i =& \dot{x}_{i0} + \Delta\dot{x}_i = f(\mathrm{x}_0 + \Delta\mathrm{x}, \mathrm{u}_0 + \Delta\mathrm{u}) \\ & f_i(\mathrm{x}_0, \mathrm{u}_0) + \frac{\partial f_i}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f_i}{\partial x_n}\Delta x_n \\ & + \frac{\partial f_i}{\partial u_1}\Delta u_1 + \cdots + \frac{\partial f_i}{\partial u_r}\Delta u_r\end{aligned}$$

Since $\dot{x}_{i0} = f_i(\mathrm{x}_0, \mathrm{u}_0)$,

$$\Delta\dot{x}_i = \frac{\partial f_i}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial f_i}{\partial x_n}\Delta x_n + \frac{\partial f_i}{\partial u_1}\Delta u_1 + \cdots + \frac{\partial f_i}{\partial u_r}\Delta u_r$$

with $i = 1, 2, \cdots, n$. Similarly,

$$\Delta y_j = \frac{\partial g_j}{\partial x_1}\Delta x_1 + \cdots + \frac{\partial g_j}{\partial x_n}\Delta x_n + \frac{\partial g_j}{\partial u_1}\Delta u_1 + \cdots + \frac{\partial g_j}{\partial u_r}\Delta u_r$$

with $j = 1, 2, \cdots, m$.

Therefore, the linearized forms of equations (3) and (4) are

$$\Delta \dot{x} = A\Delta x + B\Delta u \tag{8}$$
$$\Delta y = C\Delta x + D\Delta u \tag{9}$$

where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_r} \end{bmatrix}$$
$$C = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \quad D = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial u_1} & \cdots & \frac{\partial g_m}{\partial u_r} \end{bmatrix} \tag{10}$$

$$
\begin{aligned}
\Delta \mathrm{x} &= \text{ state vector of dimension } n \\
\Delta \mathrm{y} &= \text{ output vector of dimension } m \\
\Delta \mathrm{u} &= \text{ input vector of dimension } r \\
\mathrm{A} &= \text{ state or plant matrix of size } n \times n \\
\mathrm{B} &= \text{ control or input matrix of size } n \times n \\
\mathrm{C} &= \text{ output matrix of size } m \times n \\
\mathrm{D} &= \text{ feed forward matrix of size } m \times r
\end{aligned}
$$

By taking the Laplace transform of the equations (8) and (9),

$$
s\Delta \mathrm{x}(\mathrm{s}) - \Delta \mathrm{x}(0) = \mathrm{A}\Delta \mathrm{x}(\mathrm{s}) + \mathrm{B}\Delta \mathrm{u}(\mathrm{s}) \tag{11}
$$

$$
\Delta \mathrm{y}(\mathrm{s}) = \mathrm{C}\Delta \mathrm{x}(\mathrm{s}) + \mathrm{D}\Delta \mathrm{u}(\mathrm{s}) \tag{12}
$$

Rearranging the equation (11),

$$(sI - A)\Delta x(s) = \Delta x(0) + B\Delta u(s)$$

Hence,

$$\Delta x(s) = (sI - A)^{-1}[\Delta x(0) + B\Delta u(s)] \tag{13}$$

and

$$\Delta y(s) = C(sI - A)^{-1}[\Delta x(0) + B\Delta u(s)] + D\Delta u(s) \tag{14}$$

The poles of $\Delta x(s)$ and $\Delta y(s)$ are the roots of the equation.

$$\det(sI - A) = 0 \tag{15}$$

The values of s which satisfy the above equation are known as the *eigenvalues* of matrix A. Equation (15) is referred to as the *characteristic* equation of A.

Eigenvalues

The eigenvalues of a matrix are given by the values of the scalar parameter λ for which there exist non-trivial solutions to the following equation.

$$\mathsf{A}\phi = \lambda\phi \tag{16}$$

where

$$\mathsf{A} \text{ is an } n \times n \text{ matrix}$$
$$\phi \text{ is an } n \times 1 \text{ vector}$$

To find the eigenvalues,

$$(\mathsf{A} - \lambda\mathsf{I})\phi = 0 \tag{17}$$

For a non-trivial solution,

$$|\mathsf{A} - \lambda\mathsf{I}| = 0 \tag{18}$$

- The eigenvalues may be real or complex.
- If A is real, complex eigenvalues always occur in conjugate pairs.
- Similar matrices have identical eigenvalues.
- A matrix and its transpose have the same eigenvalues.

Eigenvectors

For any eigenvalue λ_i,

$$A\phi_{\mathrm{i}} = \lambda_i \phi_i \quad i = 1, 2, \cdots, n \tag{19}$$

The eigenvector ϕ_{i} is the right eigenvector and has the form

$$\phi_{\mathrm{i}} = \begin{bmatrix} \phi_{1i} \\ \phi_{2i} \\ \cdots \\ \phi_{ni} \end{bmatrix}$$

Similarly, the n-row vector ψ_{i} which satisfies

$$\psi_{\mathrm{i}} A = \lambda_i \psi_i \quad i = 1, 2, \cdots, n \tag{20}$$

is called the left eigenvector for the eigenvalue λ_i.

- The left and right eigenvectors corresponding to different eigenvalues are orthogonal.

$$\psi_{\mathrm{j}}\phi_{\mathrm{i}} = 0$$

- For the same eigenvalue,

$$\psi_{\mathrm{i}}\phi_{\mathrm{i}} = C_i$$

  where C_i is a non-zero constant.
- If the eigenvectors are normalized,

$$\psi_{\mathrm{i}}\phi_{\mathrm{i}} = 1$$

Let the left and right eigenvector matrices be defined as

$$\begin{aligned} \Phi &= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix} \\ \Psi &= \begin{bmatrix} {\psi_1}^T & {\psi_2}^T & \cdots & {\psi_n}^T \end{bmatrix}^T \end{aligned} \tag{21}$$

From the properties of eigenvectors,

$$\Phi\Psi = I \tag{22}$$

$$\Psi = \Phi^{-1} \tag{23}$$

Let an n dimensional diagonal matrix of eigenvalues be defined as

$$\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \tag{24}$$

In terms of these matrices, the equation (19) may be expressed as follows:

$$A\Phi = \Phi\Lambda \tag{25}$$

The above equation can be written as

$$\Lambda = \Phi^{-1}A\Phi = \Psi A\Phi \tag{26}$$

Φ, Ψ and Λ are of size $n \times n$ matrices and are called as modal matrices.

Stability of a Homogeneous System

Let us take a homogeneous system which does not have an external input. Therefore, the response is a natural response.

$$\Delta \dot{x} = A \Delta x \tag{27}$$

- Since there is cross coupling between states, it is difficult to isolate those parameters that influence the response in a significant way.
- To eliminate the cross coupling, let us consider a new state vector z.

$$\Delta x = \Phi z \tag{28}$$

where Φ is the modal matrix containing right eigenvectors of A as its columns. From (27) and (28),

$$\Phi \dot{z} = A \Phi z \tag{29}$$

The new state equation can be written as

$$\dot{z} = \Phi^{-1} A \Phi z \tag{30}$$

In view of (26), the above equation becomes

$$\dot{z} = \Lambda z \tag{31}$$

The important difference between (27) and (31) is that Λ is a diagonal matrix whereas A, in general, is non-diagonal.
Equation (31) is a set of decoupled first order equations.

$$\dot{z}_i = \lambda_i z_i \quad i = 1, 2, \cdots, n \tag{32}$$

The above equation is a simple first order differential equation whose solution is

$$z_i(t) = z_i(0) e^{\lambda_i t} \tag{33}$$

where $z_i(0)$ is the initial value of z_i.

From (33), the stability of the system can be assessed as follows:

1. If all the eigen values are on the left half of s-plane that is the real part of the eigenvalues should be having a negative value then the system is stable. This is called as asymptotic stability.
2. If at least one of the eigen values is on the right side of s-plane that is the real part of the eigen value is positive then the system is unstable. This is called as aperiodic instability.
3. The eigen values of a system with a real state matrix have complex conjugate pairs of eigen values. If at least one of the complex conjugate pair lies on the imaginary axis of the s-plane then the system becomes oscillatory. This is called as oscillatory instability.
4. If at least one eigenvalues is on the origin then the system stability cannot be assessed.

From (28), the response in terms of the original state vector is

$$\begin{aligned}\Delta \mathrm{x}(\mathrm{t}) &= \Phi \mathrm{z}(\mathrm{t}) \\ &= \begin{bmatrix}\phi_1 & \phi_2 & \cdots & \phi_\mathrm{n}\end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t)\end{bmatrix}\end{aligned} \tag{34}$$

From (33),

$$\Delta \mathrm{x}(\mathrm{t}) = \sum_{i=1}^{n} \phi_\mathrm{i} z_i(0) e^{\lambda_i t} \tag{35}$$

From (34), we have

$$\begin{aligned}\mathrm{z}(\mathrm{t}) &= \Phi^{-1}\Delta \mathrm{x}(\mathrm{t}) \\ &= \Psi \Delta \mathrm{x}(\mathrm{t})\end{aligned} \tag{36}$$

Therefore

$$z_i(t) = \psi_{\mathrm{i}}\Delta \mathrm{x}(\mathrm{t}) \tag{37}$$

With $t = 0$,

$$z_i(0) = \psi_{\mathrm{i}}\Delta \mathrm{x}(0) \tag{38}$$

Equation (35) can be written as

$$\Delta \mathrm{x}(\mathrm{t}) = \sum_{i=1}^{n} \phi_{\mathrm{i}} c_i e^{\lambda_i t} \tag{39}$$

where $c_i = \psi_{\mathrm{i}}\Delta \mathrm{x}(0)$. The time response of i^{th} state is given by

$$\Delta x_i(t) = \phi_{i1} c_1 e^{\lambda_1 t} + \phi_{i2} c_2 e^{\lambda_2 t} + \cdots + \phi_{in} c_n e^{\lambda_n t} \tag{40}$$

$$\Delta x(t) = \Phi z(t)$$
$$= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix} z(t)$$

and

$$z(t) = \Psi \Delta x(t)$$
$$= \begin{bmatrix} {\psi_1}^T & {\psi_2}^T & \cdots & {\psi_n}^T \end{bmatrix}^T \Delta x(t)$$

- $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$ are the original state variables. $z_1, z_2, \cdots, z_n$ are the transformed state variables.
- The right eigenvector gives the mode shape, i.e., the relative activity of the state variables when a particular mode is excited.
- The left eigenvector identifies which combination of the original state variables displays only the i^{th} mode.

The combined effect of activity and contribution of a state variable in a particular mode can be represented as a matrix called as participation factor. The participation factor matrix is given as

$$\mathrm{P} = \begin{bmatrix} \mathrm{p_1} & \mathrm{p_2} & \cdots & \mathrm{p_n} \end{bmatrix}$$

where

$$\mathrm{p_i} = \begin{bmatrix} p_{1i} \\ p_{2i} \\ \vdots \\ p_{ni} \end{bmatrix} = \begin{bmatrix} \phi_{1i}\psi_{i1} \\ \phi_{2i}\psi_{i2} \\ \vdots \\ \phi_{ni}\psi_{in} \end{bmatrix}$$

- The element $p_{ki} = \phi_{ki}\psi_{ik}$ is termed as the participation factor. It is a measure of k^{th} state variable in the i^{th} mode and vice versa.
- Since the eigenvectors are normalized,

$$\sum_{i=1}^{n} p_{ki} = 1; \quad \sum_{k=1}^{n} p_{ki} = 1$$

Controllability and Observability

Equations (8) and (9) are

$$\Delta \dot{x} = A\Delta x + B\Delta u$$
$$\Delta y = C\Delta x + D\Delta u$$

Expressing the above equations in terms of the transformed state variables z,

$$\Phi \dot{z} = A\Phi z + B\Delta u$$
$$\Delta y = C\Phi z + D\Delta u$$

The state equations in the decoupled form can be written as

$$\dot{z} = \Lambda z + B'\Delta u \qquad (41)$$
$$\Delta y = C'z + D\Delta u \qquad (42)$$

where

$$B' = \Phi^{-1}B \tag{43}$$
$$C' = C\Phi \tag{44}$$

- if the i^{th} row of B' is zero, the inputs have no effect on the i^{th} mode. Hence, the i^{th} mode is uncontrollable.
- if the i^{th} column of C' is zero, z_i does not contribute to the formation of outputs. Hence, the i^{th} mode is unobservable.
- B' of size $n \times r$ is referred to as the mode controllability matrix.
- C' of size $m \times n$ is referred to as the mode observability matrix.

Single Machine Connected to an Infinite Bus (SMIB)

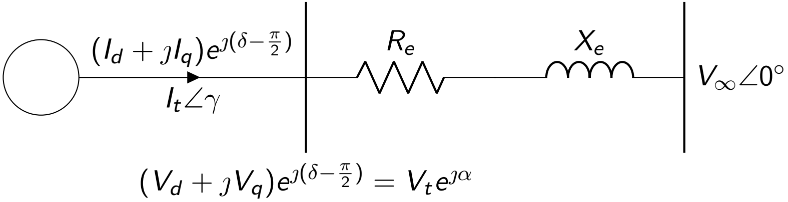


Figure: SMIB

Small Signal Stability of a SMIB

Let us analyze with varying degrees of models.

1. Constant flux linkage model - Classical Model
2. Flux decay model - Without AVR
3. Flux decay model - With AVR
4. Flux decay model - With AVR and PSS

Small Signal Stability of a SMIB - Classical Model

- Let the input mechanical torque T_m be constant.
- Therefore, turbine and speed governor dynamics are not needed.
- Flux linkages are assumed to constant. Hence, field and exciter dynamics are neglected.
- Let the generator be represented using the classical model.

The differential equations of this model are

$$\frac{d\delta}{dt} = \omega - \omega_s \tag{45}$$

$$\frac{2H}{\omega_s}\frac{d\omega}{dt} = T_m - T_e - D(\omega - \omega_{base}) \tag{46}$$

The equivalent circuit of a SMIB system using this model is shown here.

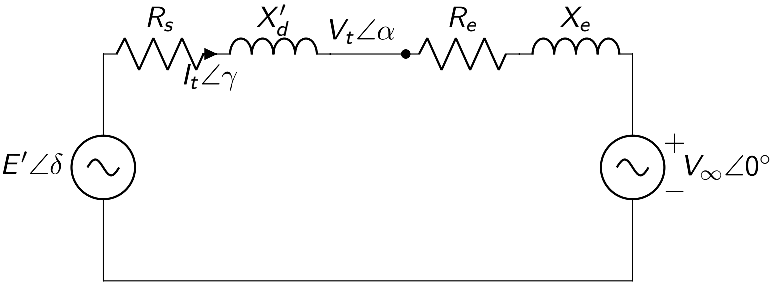


Figure: SMIB equivalent circuit using the classical model

With resistances neglected, the air-gap power (P_e) is equal to the terminal power (P_t). In per unit,

$$T_e = P_e = \frac{E' V_\infty \sin\delta}{X_T} \tag{47}$$

where $X_T = X'_d + X_e$. Linearizing around the equilibrium (δ_0) for a small disturbance gives

$$\frac{d\Delta\delta}{dt} = \Delta\omega \tag{48}$$

$$\frac{2H}{\omega_s}\frac{d\Delta\omega}{dt} = -\Delta T_e - D\Delta\omega \tag{49}$$

From (47),

$$\Delta T_e = \frac{E' V_\infty \cos\delta_0}{X_T}\Delta\delta = K_s\Delta\delta \tag{50}$$

where $K_s = \frac{E' V_\infty \cos\delta_0}{X_T}$ is the synchronizing coefficient. Expressing them in the state space form

$$\begin{bmatrix} \Delta\dot{\delta} \\ \Delta\dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{\omega_s}{2H}K_s & -\frac{\omega_s}{2H}D \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta\omega \end{bmatrix} \tag{51}$$

On taking the Laplace Transform and finding the characteristics equation,

$$s^2 + \frac{\omega_s D}{2H}s + \frac{\omega_s K_s}{2H} = 0 \tag{52}$$

By comparing it with the general form, $s^2 + 2\zeta\omega_n s + \omega_n^2$, the undamped natural frequency is

$$\omega_n = \sqrt{\frac{\omega_s K_s}{2H}} \quad \text{rad/s} \tag{53}$$

and the damping ratio is

$$\zeta = \frac{1}{2}\frac{\omega_s D}{2H\omega_n} = \frac{1}{2}\frac{D\sqrt{\omega_s}}{\sqrt{2HK_s}} \tag{54}$$

The time responses of the state variables are

$$\Delta\delta(t) = \phi_{11} c_1 e^{\lambda_1 t} + \phi_{12} c_2 e^{\lambda_2 t} \quad \text{rad} \tag{55}$$

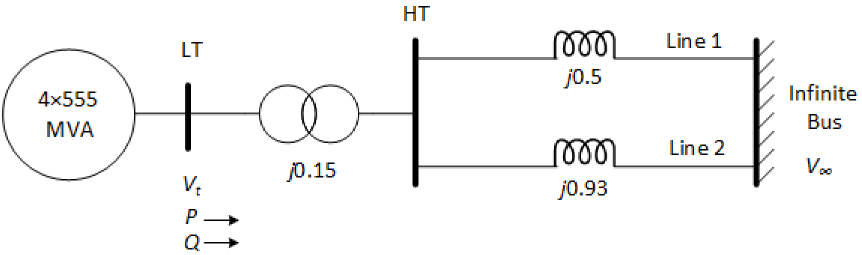
$$\Delta\omega(t) = \phi_{21} c_1 e^{\lambda_1 t} + \phi_{22} c_2 e^{\lambda_2 t} \quad \text{rad/s} \tag{56}$$

where ϕ is the right eigenvector matrix of the system matrix A and λ_1 and λ_2 are eigenvalues of it. c_1 and c_2 are calculated as follows:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \psi \begin{bmatrix} \Delta\delta(0) \\ \Delta\omega(0) \end{bmatrix} \tag{57}$$

ψ is the left eigenvector matrix of A

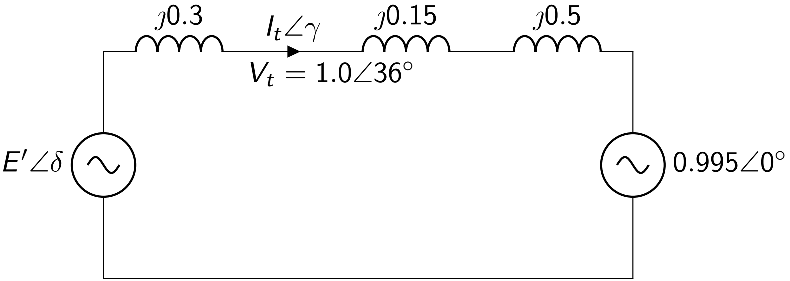
Example 12.2 from P. Kundur : Figure show the system representation applicable to a thermal generating station consisting of four 555 MVA, 24 kV, 60 Hz units.



The reactances are in per unit on 2220 MVA, 24 kV base on LT side. Analyze the small signal stability characteristics of the system about the stead operating condition following the loss of line 2. The post fault system condition in per unit on the 2220 MVA, 24 kV base is as follows:

$$P = 0.9 \quad Q = 0.3 \quad V_t = 1.0\angle 36^\circ \quad V_\infty = 0.995\angle 0^\circ$$

Use the classical model. $X'_d = 0.3$, $H = 3.5$ s and $D = 0$. Determine the time response if at $t = 0$, $\Delta\delta = 5^\circ$ and $\Delta\omega = 0$



$$\bar{I}_t = \frac{S^*}{V^*} = \frac{0.9 - \jmath 0.3}{1.0\angle -36^\circ} = 0.95\angle 17.56^\circ$$

$$\bar{E}' = \bar{V}_t + \bar{I}_t \times \jmath 0.3 = 1.0\angle 36^\circ + 0.95\angle 17.56^\circ \times \jmath 0.3 = 1.123\angle 49.91^\circ$$

The angle of E' with respect to the infinite bus is

$$\delta_0 = 49.91^\circ$$

The synchronizing torque coefficient is

$$K_s = \frac{E'V_\infty}{X_T}\cos\delta_0 = \frac{1.123 \times 0.995}{0.95}\cos 49.91^\circ = 0.7574$$

The system matrix A from the linearized equations is

$$\mathsf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{\omega_s}{2H}K_s & -\frac{\omega_s}{2H}D \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -40.79 & 0 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1, \lambda_2 = \pm \jmath 6.39$$

The right eigenvector modal matrix is

$$\Phi = \begin{bmatrix} -\jmath 0.1547 & +\jmath 0.1547 \\ 0.99 & 0.99 \end{bmatrix}$$

The left eigenvector modal matrix is

$$\Psi = \Phi^{-1} = \begin{bmatrix} \jmath 3.2323 & 0.5061 \\ -\jmath 3.2323 & 0.5061 \end{bmatrix}$$

The participation matrix is

$$P = \begin{bmatrix} \phi_{11}\psi_{11} & \phi_{12}\psi_{21} \\ \phi_{21}\psi_{12} & \phi_{22}\psi_{22} \end{bmatrix} = \begin{bmatrix} 0.5000 & 0.5000 \\ 0.5000 & 0.5000 \end{bmatrix}$$

The time response is

$$\Delta\delta(t) = \phi_{11}c_1 e^{\lambda_1 t} + \phi_{12}c_2 e^{\lambda_2 t} \quad \text{rad}$$
$$\Delta\omega(t) = \phi_{21}c_1 e^{\lambda_1 t} + \phi_{22}c_2 e^{\lambda_2 t} \quad \text{rad/s}$$

c_1 and c_2 are calculated as follows:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \psi \begin{bmatrix} \Delta\delta(0) \\ \Delta\omega(0) \end{bmatrix} = \begin{bmatrix} \jmath 3.2323 & 0.5061 \\ -\jmath 3.2323 & 0.5061 \end{bmatrix} \begin{bmatrix} 0.0873 \\ 0 \end{bmatrix} = \begin{bmatrix} \jmath 0.2822 \\ -\jmath 0.2822 \end{bmatrix}$$

The time response of rotor angle deviation is

$$\Delta\delta(t) = 0.0873\cos(6.35t) \text{ rad}$$

The time response of speed deviation is

$$\Delta\omega(t) = -0.5644\sin(6.35t) \text{ rad/s}$$

Small Signal Stability of a SMIB - Including Field Circuit Dynamics

- Let the input mechanical torque T_m be constant.
- Therefore, turbine and speed governor dynamics are not needed.
- Exciter output voltage is constant. Hence, exciter dynamics are not needed.
- Let the generator be represented using the flux decay model.

The DAE (Differential Algebraic Equations) of the flux decay model excluding exciter dynamics are

$$T'_{d0}\frac{dE'_q}{dt} = -E'_q - (X_d - X'_d)I_d + E_{fd} \tag{58}$$

$$\frac{d\delta}{dt} = \omega - \omega_s \tag{59}$$

$$\frac{2H}{\omega_s}\frac{d\omega}{dt} = T_m - T_e - D(\omega - \omega_{base}) \tag{60}$$

$$V_q + R_s I_q = -X'_d I_d + E'_q \tag{61}$$

$$V_d + R_s I_d = X_q I_q \tag{62}$$

Where

$$\begin{aligned} T_e &= \text{Real } \{((X_q - X'_d)I_q + \jmath E'_q)(I_d + \jmath I_q)^*\} \\ &= (X_q - X'_d)I_q I_d + E'_q I_q \end{aligned}$$

From the figure SMIB,

$$(I_d + \jmath I_q)e^{\jmath(\delta - \frac{\pi}{2})} = \frac{(V_d + \jmath V_q)e^{\jmath(\delta - \frac{\pi}{2})} - V_\infty e^{\jmath \angle 0}}{R_e + \jmath X_e} \tag{63}$$

On separating the real and imaginary parts from (63),

$$I_d R_e - I_q X_e = V_d - V_\infty \sin\delta \tag{64}$$

$$I_q R_e + I_d X_e = V_q - V_\infty \cos\delta \tag{65}$$

Let the steady state operating point values of $(E'_q, \delta, E_{fd}, I_d, I_q, V_d, V_q)$ be $(E'_{q0}, \delta_0, E_{fd0}, I_{d0}, I_{q0}, V_{d0}, V_{q0})$. Linearizing equations (61), (62), (64) and (65) for a small perturbation Δ around the steady operating point and neglecting R_s give

$$\begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} = \begin{bmatrix} 0 & X_q \\ -X'_d & 0 \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta E'_q \end{bmatrix} \tag{66}$$

$$\begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} = \begin{bmatrix} R_e & -X_e \\ X_e & R_e \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} + \begin{bmatrix} V_\infty \cos \delta_0 \\ -V_\infty \sin \delta_0 \end{bmatrix} \Delta\delta \tag{67}$$

Let the steady state operating point values of $(E'_q, \delta, E_{fd}, I_d, I_q, V_d, V_q)$ be $(E'_{q0}, \delta_0, E_{fd0}, I_{d0}, I_{q0}, V_{d0}, V_{q0})$. Linearizing equations (61), (62), (64) and (65) for a small perturbation Δ around the steady operating point and neglecting R_s give

$$\begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} = \begin{bmatrix} 0 & X_q \\ -X'_d & 0 \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta E'_q \end{bmatrix} \tag{66}$$

$$\begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} = \begin{bmatrix} R_e & -X_e \\ X_e & R_e \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} + \begin{bmatrix} V_\infty \cos\delta_0 \\ -V_\infty \sin\delta_0 \end{bmatrix} \Delta\delta \tag{67}$$

By equating (66) and (67),

$$\begin{bmatrix} 0 & X_q \\ -X'_d & 0 \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta E'_q \end{bmatrix} = \begin{bmatrix} R_e & -X_e \\ X_e & R_e \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} + \begin{bmatrix} V_\infty \cos\delta_0 \\ -V_\infty \sin\delta_0 \end{bmatrix} \Delta\delta \tag{68}$$

Solving for ΔI_d and ΔI_q from (68),

$$\begin{bmatrix}\Delta I_d\\ \Delta I_q\end{bmatrix} = \begin{bmatrix} R_e & -(X_e+X_q)\\ (X_e+X_d') & R_e\end{bmatrix}^{-1}\left(\begin{bmatrix}0\\ \Delta E_q'\end{bmatrix} + \begin{bmatrix}-V_\infty\cos\delta_0\\ V_\infty\sin\delta_0\end{bmatrix}\Delta\delta\right)$$
$$= \frac{1}{R_e^2+(X_e+X_q)(X_e+X_d')}\begin{bmatrix} R_e & (X_e+X_q)\\ -(X_e+X_d') & R_e\end{bmatrix}$$
$$\begin{bmatrix}-V_\infty\cos\delta_0 & 0\\ V_\infty\sin\delta_0 & 1\end{bmatrix}\begin{bmatrix}\Delta\delta\\ \Delta E_q'\end{bmatrix}$$
$$= \frac{1}{R_e^2+(X_e+X_q)(X_e+X_d')}$$
$$\begin{bmatrix} V_\infty(X_e+X_q)\sin\delta_0 - R_eV_\infty\cos\delta_0 & (X_e+X_q)\\ V_\infty(X_e+X_d')\cos\delta_0 + R_eV_\infty\sin\delta_0 & R_e\end{bmatrix}\begin{bmatrix}\Delta\delta\\ \Delta E_q'\end{bmatrix} \tag{69}$$

On linearizing differential equations (58)-(60),

$$\begin{bmatrix}\Delta \dot{E}_q{}' \\ \Delta\dot{\delta} \\ \Delta\dot{\omega}\end{bmatrix} = \begin{bmatrix} -\frac{1}{T'_{d0}} & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{\omega_{base}}{2H} I_{q0} & 0 & -\frac{\omega_{base}}{2H} D \end{bmatrix} \begin{bmatrix}\Delta E'_q \\ \Delta\delta \\ \Delta\omega\end{bmatrix}$$
$$+ \begin{bmatrix} -\frac{1}{T'_{d0}}(X_d - X'_d) & 0 \\ 0 & 0 \\ -\frac{\omega_{base}}{2H}(X_q - X'_d) I_{q0} & -\frac{\omega_{base}}{2H}((X_q - X'_d) I_{d0} + E'_{q0}) \end{bmatrix} \begin{bmatrix}\Delta I_d \\ \Delta I_q\end{bmatrix} \tag{70}$$
$$+ \begin{bmatrix} \frac{1}{T'_{d0}} & 0 \\ 0 & 0 \\ 0 & \frac{\omega_{base}}{2H} \end{bmatrix} \begin{bmatrix}\Delta E_{fd} \\ \Delta T_m\end{bmatrix}$$

On linearizing differential equations (58)-(60),

$$\begin{bmatrix} \Delta \dot{E}_q{}' \\ \Delta \dot{\delta} \\ \Delta \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{1}{T'_{d0}} & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{\omega_{base}}{2H} I_{q0} & 0 & -\frac{\omega_{base}}{2H} D \end{bmatrix} \begin{bmatrix} \Delta E'_q \\ \Delta \delta \\ \Delta \omega \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{1}{T'_{d0}}(X_d - X'_d) & 0 \\ 0 & 0 \\ -\frac{\omega_{base}}{2H}(X_q - X'_d) I_{q0} & -\frac{\omega_{base}}{2H}((X_q - X'_d) I_{d0} + E'_{q0}) \end{bmatrix} \begin{bmatrix} \Delta I_d \\ \Delta I_q \end{bmatrix} \qquad (70)$$

$$+ \begin{bmatrix} \frac{1}{T'_{d0}} & 0 \\ 0 & 0 \\ 0 & \frac{\omega_{base}}{2H} \end{bmatrix} \begin{bmatrix} \Delta E_{fd} \\ \Delta T_m \end{bmatrix}$$

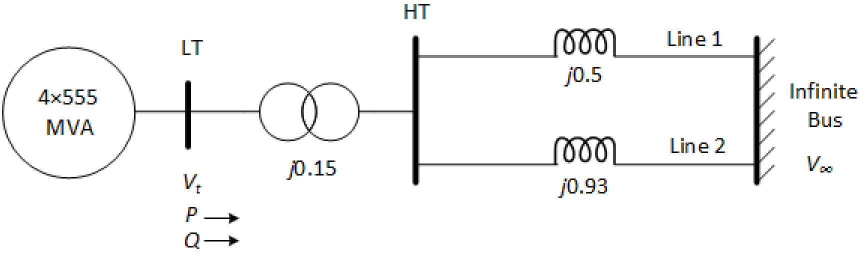
Substituting (69) in (70) and simplifying lead to

$$\begin{bmatrix} \Delta \dot{E}_q{}' \\ \Delta \dot{\delta} \\ \Delta \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{1}{K_3 T'_{d0}} & -\frac{K_4}{T'_{d0}} & 0 \\ 0 & 0 & 1 \\ -\frac{\omega_{base}}{2H} K_2 & -\frac{\omega_{base}}{2H} K_1 & -\frac{\omega_{base}}{2H} D \end{bmatrix} \begin{bmatrix} \Delta E'_q \\ \Delta \delta \\ \Delta \omega \end{bmatrix} + \begin{bmatrix} \frac{1}{T'_{d0}} & 0 \\ 0 & 0 \\ 0 & \frac{\omega_{base}}{2H} \end{bmatrix} \begin{bmatrix} \Delta E_{fd} \\ \Delta T_m \end{bmatrix} \qquad (71)$$

where

$$
\begin{aligned}
K_1 = -\frac{1}{R_e^2 + (X_e + X_q)(X_e + X'_d)} \times \\
[I_{q0} V_\infty (X'_d - X_q)((X_e + X_q)\sin\delta_0 - R_e \cos\delta_0) \\
+ V_\infty ((X'_d - X_q) I_{d0} - E'_{q0})((X_e + X'_d)\cos\delta_0 + R_e \sin\delta_0)] \\
K_2 = \frac{1}{R_e^2 + (X_e + X_q)(X_e + X'_d)} [I_{q0}((R_e^2 + (X_e + X_q)(X_e + X'_d)) - \\
(X'_d - X_q)(X_e + X_q)) - R_e (X'_d - X_q) I_{d0} + R_e E'_{q0}] \\
\frac{1}{K_3} = 1 + \frac{(X_d - X'_d)(X_e + X_q)}{R_e^2 + (X_e + X_q)(X_e + X'_d)} \\
K_4 = \frac{V_\infty (X_d - X'_d)}{R_e^2 + (X_e + X_q)(X_e + X'_d)} [(X_e + X_q)\sin\delta_0 - R_e \cos\delta_0]
\end{aligned}
\tag{72}
$$

Example 12.3 from P. Kundur : This example analyzes the small signal stability of the example 12.2 including filed circuit dynamics.



The parameters of each of the four generators of the plant are in per unit on its rating are as follows:

$$X_d = 1.81 \quad X_q = 1.76 \quad X'_d = 0.3 \quad X_l = 0.16$$

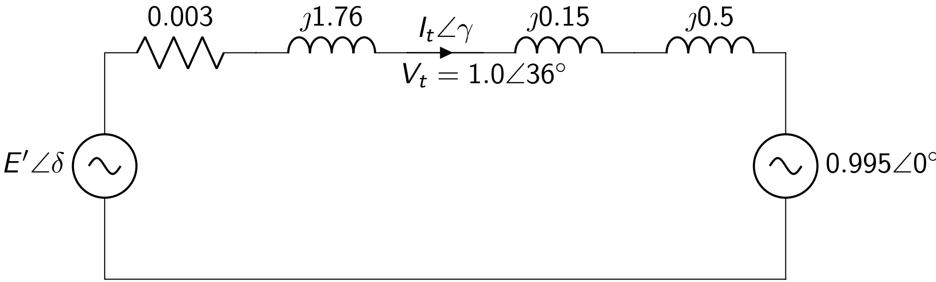
$$R_s = 0.003 \quad T'_{do} = 8 \text{ s} \quad H = 3.5 \quad D = 0$$

If the plant output is

$$P = 0.9 \quad Q = 0.3 \quad V_t = 1.0\angle 36^\circ \quad V_\infty = 0.995\angle 0^\circ$$

compute the following:

1. Eigenvalues of A and the corresponding eigenvectors and participation matrix.
2. Synchronizing torque coefficient K_s and damping torque coefficient K_d



$$\bar{I}_t = \frac{S^*}{V^*} = \frac{0.9 - \jmath 0.3}{1.0\angle{-36^\circ}} = 0.95\angle 17.56^\circ$$

$$\bar{E}' = \bar{V}_t + \bar{I}_t \times (R_s + \jmath X_q) = 1.1253\angle 49.84^\circ$$

The angle of E' with respect to the infinite bus is

$$\delta_0 = 81.99^\circ$$

$$V_{d0} = V_t \sin(\delta_0 - \alpha) = 0.7192$$

$$V_{q0} = V_t \cos(\delta_0 - \alpha) = 0.6948$$

$$I_{d0} = I_t \sin(\delta_0 - \gamma) = 0.8569$$

$$I_{q0} = I_t \cos(\delta_0 - \gamma) = 0.4101$$

$$E'_{q0} = V_{q0} + I_{q0} R_s + I_{d0} X'_d = 0.9531$$

$$E_{fd0} = E'_{q0} + (X_d - X'_d) I_d = 2.2471$$

The system matrix A is

$$\mathrm{A} = \begin{bmatrix} -0.3237 & -0.1957 & 0 \\ 0 & 0 & 1.0000 \\ -56.0058 & -40.2725 & 0 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = -0.0515; \quad \lambda_2, \lambda_3 = -0.1361 \pm \jmath 6.34$$

Small Signal Stability of a SMIB - Including Exciter Dynamics

- Let the input mechanical torque T_m be constant.
- Therefore, turbine and speed governor dynamics are not needed.
- Exciter dynamics, field flux variations along with rotor dynamics are considered.

A simplified fast acting static exciter is now considered which is represented by a first order control block with a gain K_A and time constant T_A as shown here.

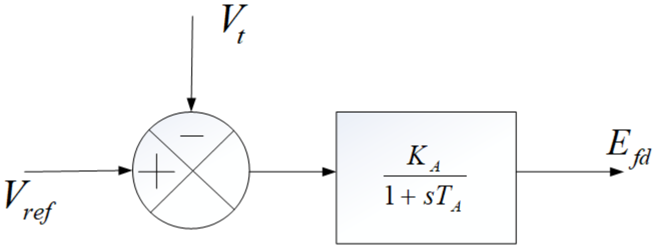


Figure: Simplified Fast Acting Static Exciter

The differential equation representing the dynamics of the exciter is

$$T_A \frac{dE_{fd}}{dt} = -E_{fd} + K_A(V_{ref} - V_t) \tag{73}$$

Linearizing (73) gives

$$T_A \frac{d\Delta E_{fd}}{dt} = -\Delta E_{fd} + K_A(\Delta V_{ref} - \Delta V_t) \tag{74}$$

The terminal voltage in terms of $dq0$ parameters is

$$V_t = \sqrt{V_d^2 + V_q^2} \tag{75}$$

Linearizing (75) gives

$$\Delta V_t = \frac{V_{d0}}{V_t}\Delta V_d + \frac{V_{q0}}{V_t}\Delta V_q = \begin{bmatrix} \frac{V_{d0}}{V_t} & \frac{V_{q0}}{V_t} \end{bmatrix} \begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} \tag{76}$$

Linearizing (73) gives

$$T_A \frac{d\Delta E_{fd}}{dt} = -\Delta E_{fd} + K_A(\Delta V_{ref} - \Delta V_t) \tag{74}$$

The terminal voltage in terms of $dq0$ parameters is

$$V_t = \sqrt{V_d^2 + V_q^2} \tag{75}$$

Linearizing (75) gives

$$\Delta V_t = \frac{V_{d0}}{V_t}\Delta V_d + \frac{V_{q0}}{V_t}\Delta V_q = \begin{bmatrix} \frac{V_{d0}}{V_t} & \frac{V_{q0}}{V_t} \end{bmatrix} \begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} \tag{76}$$

Substituting (69) in (66) gives

$$\begin{bmatrix} \Delta V_d \\ \Delta V_q \end{bmatrix} = \frac{1}{R_e^2 + (X_e + X_q)(X_e + X_d')} \begin{bmatrix} 0 & X_q \\ -X_d' & 0 \end{bmatrix}$$
$$\begin{bmatrix} V_\infty (X_e + X_q)\sin\delta_0 - R_e V_\infty \cos\delta_0 & (X_e + X_q) \\ V_\infty (X_e + X_d')\cos\delta_0 + R_e V_\infty \sin\delta_0 & R_e \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta E_q' \end{bmatrix} + \begin{bmatrix} 0 \\ \Delta E_q' \end{bmatrix} \tag{77}$$

Substituting (77) in (76) and simplifying give

$$\Delta V_t = K_5 \Delta\delta + K_6 \Delta E_q' \tag{78}$$

where

$$K_5 = \frac{1}{R_e^2 + (X_e + X_q)(X_e + X_d')}\Big(\frac{V_{d0}}{V_t} X_q (R_e V_\infty \sin\delta_0 + V_\infty (X_e + X_d')\cos\delta_0)$$
$$+ \frac{V_{q0}}{V_t} X_d' (R_e V_\infty \cos\delta_0 - V_\infty (X_e + X_q)\sin\delta_0)\Big)$$

$$K_6 = \frac{1}{R_e^2 + (X_e + X_q)(X_e + X_d')}\Big(\frac{V_{d0}}{V_t} X_q R_e - \frac{V_{q0}}{V_t} X_d' (X_e + X_q)\Big) + \frac{V_{q0}}{V_t}$$

Equation (74) can be substituted in (71), with the linearized terminal voltage defined in (78), and can be written as

$$\begin{bmatrix} \Delta \dot{E}_q{}' \\ \Delta \dot{\delta} \\ \Delta \dot{\omega} \\ \Delta \dot{E}_{fd} \end{bmatrix} = \begin{bmatrix} -\frac{1}{K_3 T'_{d0}} & -\frac{K_4}{T'_{d0}} & 0 & \frac{1}{T'_{d0}} \\ 0 & 0 & 1 & 0 \\ -\frac{\omega_{base}}{2H} K_2 & -\frac{\omega_{base}}{2H} K_1 & -\frac{\omega_{base}}{2H} D & 0 \\ -\frac{K_A K_6}{T_A} & -\frac{K_A K_5}{T_A} & 0 & -\frac{1}{T_A} \end{bmatrix} \begin{bmatrix} \Delta E'_q \\ \Delta \delta \\ \Delta \omega \\ \Delta E_{fd} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{\omega_{base}}{2H} \\ \frac{K_A}{T_A} & 0 \end{bmatrix} \begin{bmatrix} \Delta V_{ref} \\ \Delta T_m \end{bmatrix} \tag{79}$$

Equation (74) can be substituted in (71), with the linearized terminal voltage defined in (78), and can be written as

$$\begin{bmatrix} \Delta \dot{E}_q{}' \\ \Delta \dot{\delta} \\ \Delta \dot{\omega} \\ \Delta \dot{E}_{fd} \end{bmatrix} = \begin{bmatrix} -\frac{1}{K_3 T'_{d0}} & -\frac{K_4}{T'_{d0}} & 0 & \frac{1}{T'_{d0}} \\ 0 & 0 & 1 & 0 \\ -\frac{\omega_{base}}{2H} K_2 & -\frac{\omega_{base}}{2H} K_1 & -\frac{\omega_{base}}{2H} D & 0 \\ -\frac{K_A K_6}{T_A} & -\frac{K_A K_5}{T_A} & 0 & -\frac{1}{T_A} \end{bmatrix} \begin{bmatrix} \Delta E'_q \\ \Delta \delta \\ \Delta \omega \\ \Delta E_{fd} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{\omega_{base}}{2H} \\ \frac{K_A}{T_A} & 0 \end{bmatrix} \begin{bmatrix} \Delta V_{ref} \\ \Delta T_m \end{bmatrix} \tag{79}$$

- The constants K_1 and K_6 are called as Heffron-Phillips constants and the state space model is called Heffron-Phillips model.
- It can be observed from (79) that the nonlinear SMIB system has been linearized and now represented in state space with the state variables being $\begin{bmatrix} \Delta E'_q & \Delta \delta & \Delta \omega & \Delta E_{fd} \end{bmatrix}$.
- The stability of the system now depends on the eigen values of the state matrix.

The state space representation of the system given in (79) can be represented as a control block diagram as shown here.

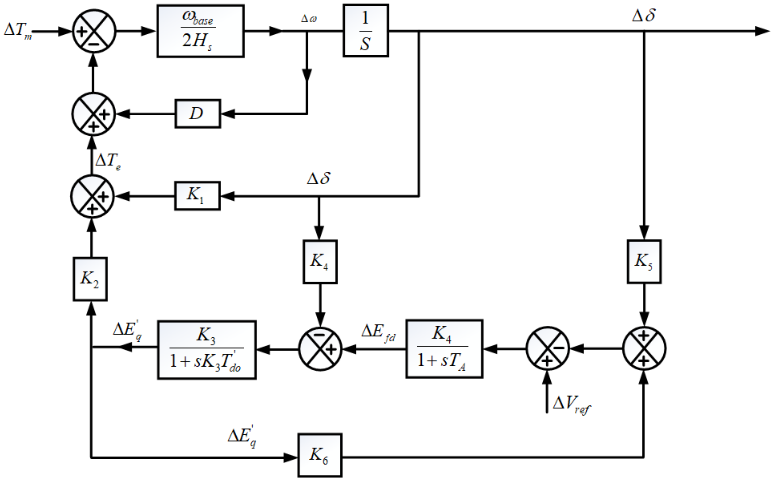


Figure: Heffron-Phillips model

Taking the Laplace transform of (79) gives

$$\begin{aligned} s\Delta E'_q(s) &= -\frac{1}{K_3 T'_{d0}}\Delta E'_q(s) - \frac{K_4}{T'_{d0}}\Delta\delta(s) + \frac{1}{T'_{d0}}\Delta E_{fd}(s) \\ s\Delta\delta(s) &= \Delta\omega(s) \\ s\Delta\omega(s) &= -\frac{\omega_{base}}{2H}K_2\Delta E'_q(s) - \frac{\omega_{base}}{2H}K_1\Delta\delta(s) - \frac{\omega_{base}}{2H}D\Delta\omega(s) \\ &\quad + \frac{\omega_{base}}{2H}\Delta T_m(s) \\ s\Delta E_{fd}(s) &= -\frac{K_A K_6}{T_A}\Delta E'_q(s) - \frac{K_A K_5}{T_A}\Delta\delta(s) - \frac{1}{T_A}\Delta E_{fd}(s) + \frac{K_A}{T_A}\Delta V_{ref}(s) \end{aligned} \tag{80}$$

Simplifying (80) gives

$$\Delta E'_q(s) = -\frac{K_4 K_3}{1 + sK_3 T'_{d0}} \Delta\delta(s) + \frac{K_3}{1 + sK_3 T'_{d0}} \Delta E_{fd}(s) \tag{81}$$

$$\Delta E_{fd}(s) = -\frac{K_A}{1 + sT_A} (K_5 \Delta\delta(s) + K_6 \Delta E'_q(s)) + \frac{K_A}{1 + sT_A} \Delta V_{ref}(s) \tag{82}$$

$$\Delta\omega(s) = s\Delta\delta(s) \tag{83}$$

$$s^2 \Delta\delta(s) = \frac{\omega_{base}}{2H} \Delta T_m(s) - \frac{\omega_{base}}{2H} (K_1 \Delta\delta(s) + K_2 \Delta E'_q(s)) - \frac{\omega_{base}}{2H} Ds\Delta\delta(s) \tag{84}$$

Simplifying (80) gives

$$\Delta E_q'(s) = -\frac{K_4 K_3}{1 + sK_3 T_{d0}'}\Delta\delta(s) + \frac{K_3}{1 + sK_3 T_{d0}'}\Delta E_{fd}(s) \tag{81}$$

$$\Delta E_{fd}(s) = -\frac{K_A}{1 + sT_A}(K_5\Delta\delta(s) + K_6\Delta E_q'(s)) + \frac{K_A}{1 + sT_A}\Delta V_{ref}(s) \tag{82}$$

$$\Delta\omega(s) = s\Delta\delta(s) \tag{83}$$

$$s^2\Delta\delta(s) = \frac{\omega_{base}}{2H}\Delta T_m(s) - \frac{\omega_{base}}{2H}(K_1\Delta\delta(s) + K_2\Delta E_q'(s)) - \frac{\omega_{base}}{2H}Ds\Delta\delta(s) \tag{84}$$

Assuming $\Delta V_{ref}(s) = 0$ and substituting (82) in (81), we get

$$\Delta E_q'(s) = -\frac{(K_4(1 + sT_A) + K_A K_5)K_3}{(1 + sK_3 T_{d0}')(1 + sT_A) + K_A K_6 K_3}\Delta\delta(s) \tag{85}$$

The linearized electrical torque ΔT_e expression is

$$\begin{aligned}\Delta T_e &= \Delta E'_q I_q + E'_q \Delta I_q + (X_q - X'_d) I_q \Delta I_d + (X_q - X'_d) I_d \Delta I_q \\ &= I_q \Delta E'_q + \begin{bmatrix}(X_q - X'_d) I_q & E'_q + (X_q - X'_d) I_d\end{bmatrix} \begin{bmatrix}\Delta I_d \\ \Delta I_q\end{bmatrix}\end{aligned} \tag{86}$$

On substituting (69) in (86), we get

$$\Delta T_e = K_1 \Delta\delta + K_2 \Delta E'_q \tag{87}$$

By taking the Laplace transform of (87) and substituting (85) in it,

$$\Delta T_e(s) = H(s) \Delta\delta(s) \tag{88}$$

where

$$H(s) = K_1 - \frac{(K_4(1 + sT_A) + K_A K_5) K_3 K_2}{(1 + sK_3 T'_{d0})(1 + sT_A) + K_A K_6 K_3}$$

On substituting (88) in (84) and assuming $\Delta T_m(s) = 0$, we get

$$\left(s^2 + \frac{\omega_{base}}{2H} Ds + \frac{\omega_{base}}{2H} H(s)\right) \Delta\delta(s) = 0 \tag{89}$$

- The small signal oscillations are typically in the range of 0.1 Hz to 3 Hz.
- In this region of frequency interest, the transfer function $H(s)$ can be split into two components.

$$\begin{aligned} H(s = \jmath\omega) &= K_1 - \frac{(K_4(1 + \jmath\omega T_A) + K_A K_5) K_3 K_2}{(1 + \jmath\omega K_3 T'_{d0})(1 + \jmath\omega T_A) + K_A K_6 K_3} \\ &= K_1 - \frac{(K_4(1 + \jmath\omega T_A) + K_A K_5) K_3 K_2}{(1 + K_A K_6 K_3 - \omega^2 K_3 T_A T'_{d0}) + \jmath\omega(K_3 T'_{d0} + T_A)} \\ &= K_1 - \frac{K_2((K_4 + K_A K_5) + \jmath\omega K_4 T_A)}{(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0}) + \jmath\omega(T'_{d0} + \frac{T_A}{K_3})} \end{aligned} \tag{90}$$

Neglecting the effect of constant K_4

$$H(\jmath\omega) = K_1 - \frac{K_2 K_A K_5}{(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0}) + \jmath\omega(T'_{d0} + \frac{T_A}{K_3})}$$

$$= \underbrace{K_1 - \frac{K_2 K_A K_5 \left(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0}\right)}{\left(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0}\right)^2 + \left(\omega(T'_{d0} + \frac{T_A}{K_3})\right)^2}}_{K_s} \tag{91}$$

$$+ \jmath\omega \underbrace{\frac{K_2 K_A K_5 \left(T'_{d0} + \frac{T_A}{K_3}\right)}{\left(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0}\right)^2 + \left(\omega(T'_{d0} + \frac{T_A}{K_3})\right)^2}}_{K_d}$$

$$H(s) = K_s + sK_d \tag{92}$$

The electrical torque is

$$\begin{aligned}\Delta T_e(\jmath\omega) &= K_s\Delta\delta + \jmath\omega K_d\Delta\delta \\ &= K_s\Delta\delta + K_d\Delta\omega\end{aligned} \tag{93}$$

- It has two components.
- The synchronising torque K_s is in phase with the change in rotor angle $\Delta\delta$.
- The damping torque K_d is in quadrature to the rotor angle or in phase with the change in speed $\Delta\omega$.

Substituting (79) in (76) gives

$$\left(s^2 + \frac{\omega_{base}}{2H}(K_d + D)s + \frac{\omega_{base}}{2H}K_s\right)\Delta\delta(s) = 0 \tag{94}$$

- Equation (94) is a damped second order homogeneous system.
- If the damping coefficient $(D+K_d)$ were zero, the system would become undamped with natural frequency of oscillations being

$$\omega_n = \pm \jmath \sqrt{\frac{\omega_{base}}{2H} K_s} \tag{95}$$

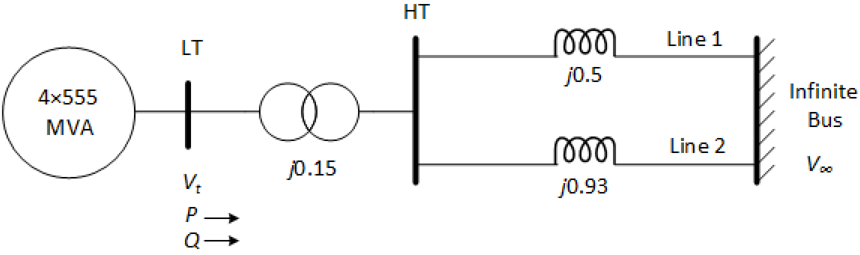
- In the undamped system, if K_s is zero for a forced system that is $\Delta T_m \neq 0$, the system will become aperiodically unstable that is the rotor angle keeps on increasing and ultimately leading to loss of synchronism.
- In case, $(K_d + D)$ becomes negative due to system conditions, there will be oscillatory instability that is the peak of the oscillations keep on increasing leading to loss of synchronism.
- If both K_s and K_d are positive for a given system, the system will settle down to a steady state operating point with oscillations being damped out after a disturbance.

$$
\begin{aligned}
K_s &= K_1 - \frac{K_2 K_A K_5 \left(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0} \right)}{\left(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0} \right)^2 + \left(\omega (T'_{d0} + \frac{T_A}{K_3}) \right)^2} \\
K_d &= \frac{K_2 K_A K_5 \left(T'_{d0} + \frac{T_A}{K_3} \right)}{\left(\frac{1}{K_3} + K_A K_6 - \omega^2 T_A T'_{d0} \right)^2 + \left(\omega (T'_{d0} + \frac{T_A}{K_3}) \right)^2}
\end{aligned} \tag{96}
$$

From the above equation, the following observations are made.

- If K_5 is negative, K_s will be positive and K_d will be negative. Hence, negative damping and oscillatory instability.
- This effect of negative damping torque due to negative K_5 gets pronounced if the exciter has a very high gain K_A.
- In a practical system, K_5 can become negative in a heavily loaded case and a high exciter gain K_A can lead to stability issues.
- The constant K_5 becomes important while assessing the stability.

Example 12.4 from P. Kundur : This example analyzes the small signal stability of the example 12.3 including exciter dynamics.



The parameters of each of the four generators of the plant are in per unit on its rating are as follows:

$$X_d = 1.81 \quad X_q = 1.76 \quad X_d' = 0.3 \quad X_l = 0.16$$

$$R_s = 0.003 \quad T_{do}' = 8 \text{ s} \quad H = 3.5 \quad D = 0$$

The exciter parameters are

$$K_A = 200 \quad T_A = 0.02$$

If the plant output is

$$P = 0.9 \quad Q = 0.3 \quad V_t = 1.0\angle 36^\circ \quad V_\infty = 0.995\angle 0^\circ$$

compute the following:

1. Eigenvalues of A.

The constants $K1$ to $K6$ are

$$K1 = 0.7478;\ K2 = 1.0399;\ K3 = 0.3862$$

$$K4 = 1.5660;\ K5 = -0.1432;\ K6 = 0.4756$$

The system matrix A is

$$A = \begin{bmatrix} -0.3 & -0.2 & 0 & 0.1 \\ 0 & 0 & 1 & 0 \\ -56 & -40.3 & 0 & 0 \\ -4756 & 1432.4 & 0 & -50 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = -28.4666 \quad \lambda_2 = -22.8857 \quad \lambda_3, \lambda_4 = 0.5143 \pm \jmath 7.2137$$

For $K_A = 10$, the eigenvalues are

$$\lambda_1 = -49.3901\ \lambda_2 = -0.9110;\ \lambda_3, \lambda_4 = -0.0113 \pm \jmath 6.3287$$

Power System Stabilizer (PSS)

- Power system stabilizer is a device used for overcoming the negative damping effect of the high gain exciter.
- It acts as a supplementary controller to the excitation system.
- Inputs to the power system stabilizer can be change in frequency, speed, power or a combination of these.
- The output is a voltage signal introduced in the excitation system to control the output of the exciter.
- The basic idea of the power system stabilizer is to introduce a pure damping term in (94) so as to counter the negative damping effect of the exciter.

- Let PSS have a transfer function $G(s)$ and the change in the speed be the input to the PSS.
- The output of the PSS be ΔV_{PSS}.
- The output of PSS is added as a supplementary signal to the exciter reference.

Therefore, the equation (74) becomes

$$T_A \frac{d\Delta E_{fd}}{dt} = -\Delta E_{fd} + K_A(\Delta V_{ref} + \Delta V_{PSS} - \Delta V_t) \tag{97}$$

The transfer function between ΔV_{PSS} and ΔT_e is found to see the effect of ΔV_{PSS} on K_s and K_d.

To get the transfer function, let us assume that $\Delta\delta = 0$ and $\Delta V_{ref} = 0$. From (81) and (82),

$$\Delta E_q'(s) = \frac{K_3}{1 + sK_3 T_{d0}'} \Delta E_{fd}(s) \tag{98}$$

$$\Delta E_{fd}(s) = -\frac{K_A K_6}{1 + sT_A} \Delta E_q'(s) + \frac{K_A}{1 + sT_A} \Delta V_{PSS}(s) \tag{99}$$

Substituting (99) in (98) gives

$$\Delta E_q'(s) = \frac{K_3 K_A}{(1 + sK_3 T_{d0}')(1 + sT_A) + K_3 K_A K_6} \Delta V_{PSS}(s) \tag{100}$$

Substituting (100) in the Laplace transform of (100) gives

$$\Delta T_e(s) = K_2 \Delta E_q'(s) = \frac{K_2 K_3 K_A}{(1 + sK_3 T_{d0}')(1 + sT_A) + K_3 K_A K_6} \Delta V_{PSS}(s) \tag{101}$$

Equation (101) can be approximated for the usual range of constants as follows

$$\Delta T_e(s) = \frac{K_2 K_A}{(1 + \frac{sT'_{d0}}{K_A K_6})(1 + sT_A)(\frac{1}{K_3} + K_A K_6)} \Delta V_{PSS}(s) \tag{102}$$

For a high gain exciter, $\frac{1}{K_3} << K_A K_6$.

$$\Delta T_e(s) = \frac{\frac{K_2}{K_6}}{(1 + \frac{sT'_{d0}}{K_A K_6})(1 + sT_A)} \Delta V_{PSS}(s) \tag{103}$$

Since PSS has a transfer function $G(s)$ with input $\Delta\omega$, Equation (103) can be written as

$$\Delta T_e(s) = \frac{\frac{K_2}{K_6}}{(1 + \frac{sT'_{d0}}{K_A K_6})(1 + sT_A)} G(s)\Delta\omega \tag{104}$$

If PSS is to contribute pure damping throughout the frequency of interest, the transfer function between ΔT_e and $\Delta\omega$ should be

$$G(s) = K_{PSS}(1 + \frac{sT'_{d0}}{K_A K_6})(1 + sT_A)$$

where K_{PSS} is the gain of PSS.

$$\Delta T_e = \frac{K_2}{K_6} K_{PSS} \Delta\omega = \frac{K_2}{K_6} K_{PSS} s \Delta\delta \qquad (105)$$

With the help of (105), the equation (94) can be approximated as

$$\left(s^2 + \frac{\omega_{base}}{2H}(K_d + D + \frac{K_2}{K_6} K_{PSS})s + \frac{\omega_{base}}{2H} K_s)\right) \Delta\delta(s) = 0 \qquad (106)$$

Hence, PSS improves the damping of system by providing a positive damping torque.

- Since $G(s)$ is a pure lead function, it is not physically realizable.
- Hence, we have a compromise resulting in what is called a lead-lag type transfer function such that it provides enough phase lead over the expected range of frequencies.

$$G(s) = K_{PSS} \frac{sT_w}{1 + sT_w} \left(\frac{1 + sT_1}{1 + sT_2} \right)^n \tag{107}$$

where

K_{PSS} is the gain.

T_w is the washout filter time constant.

T_1, T_2 are the lead-lag network time constants.

n is the number of lead-lag network blocks.

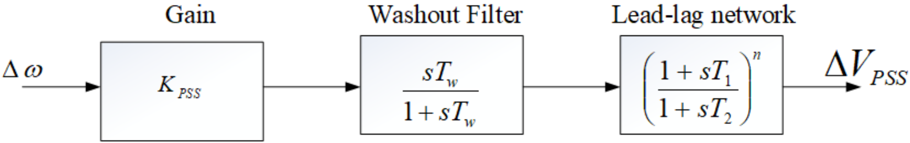


Figure: Block diagram of PSS

The procedure for finding the parameters K_{PSS}, T_1 and T_2 is as follows:
Let

$$G_{EX}(s) = \frac{K_2K_3K_A}{(1+sK_3T'_{d0})(1+sT_A)+K_3K_AK_6} \tag{108}$$

1. Find the natural frequency of oscillations of the system without damping. The flux linkage can be considered constant so that $\Delta E_q'$ is zero.

$$\Delta T_e = K_1 \Delta\delta$$

Also the damping coefficient D is assumed to be zero. With these assumptions, the equation (94) will become

$$\left(s^2 + \frac{\omega_{base}}{2H} K_1)\right) \Delta\delta(s) = 0 \tag{109}$$

$$\omega_n = \pm \jmath \sqrt{\frac{\omega_{base}}{2H} K_1} \tag{110}$$

2. At the natural frequency, find the time constant of lead-lag network such that the lag due to $G_{EX}(s)$ is compensated.

$$\angle G(\jmath\omega_n) + \angle G_{EX}(\jmath\omega_n) = 0 \tag{111}$$

3. For a high value of washout time constant, the washout filter will not contribute any angle to the transfer function of PSS hence for $n = 1$.

$$\angle(1 + \jmath\omega_n T_1) - \angle(1 + \jmath\omega_n T_2) + \angle G_{EX}(\jmath\omega_n) = 0 \tag{112}$$

   One of the lead-lag time constants can be arbitrarily chosen. The other time constant can be derived from (112).

4. To compute K_{PSS}, we can compute K^*_{PSS} i.e., the gain at which the system becomes unstable using the root locus, and then have $K_{PSS} = \frac{1}{3}K^*_{PSS}$.
5. The PSS should be activated only when low-frequency oscillations develop and should be automatically terminated when the system oscillation ceases. It should not interfere with the regular function of the excitation system during steady state operation of the system frequency. The washout stage has the transfer function

$$G_w(s) = \frac{sT_W}{1 + sT_W} \tag{113}$$

Washout filter acts like a high-pass filter allowing those signals which are above the cut-off frequency. The washout filter time constant is chosen so that the signal with lowest frequency of interest can be passed through. For washout filter time constant of $T_w = 10$ s, signals of frequency 0.1 Hz and above can be passed. The stead state operation is considered as DC signals and hence blocked by the washout filter.

- Let the output of the washout filter be ΔV_{WF}.
- Let the output of the lead-lag block be ΔV_{PSS}.

The linearized dynamic equations of the PSS are

$$\Delta \dot{V}_{WF} = -\frac{1}{T_W}\Delta V_{WF} + K_{PSS}\Delta\dot{\omega} \tag{114}$$

$$\Delta \dot{V}_{PSS} = -\frac{1}{T_2}\Delta V_{PSS} + \frac{1}{T_2}\Delta V_{WF} + \frac{T_1}{T_2}\Delta \dot{V}_{WF} \tag{115}$$

By substituting $\Delta\dot{\omega}$ from (79) in (114) and by substituting the resultant equation in (115), we get

$$\Delta\dot{V}_{WF} = \begin{bmatrix} -\frac{\omega_{base}}{2H}K_2K_{PSS} & -\frac{\omega_{base}}{2H}K_1K_{PSS} & -\frac{\omega_{base}}{2H}DK_{PSS} & 0 & -\frac{1}{T_W} & 0 \end{bmatrix} \begin{bmatrix} \Delta E_q' \\ \Delta\delta \\ \Delta\omega \\ \Delta E_{fd} \\ \Delta V_{WF} \\ \Delta V_{PSS} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\omega_{base}}{2H}K_{PSS} \end{bmatrix} \begin{bmatrix} \Delta V_{ref} \\ \Delta T_m \end{bmatrix} \tag{116}$$

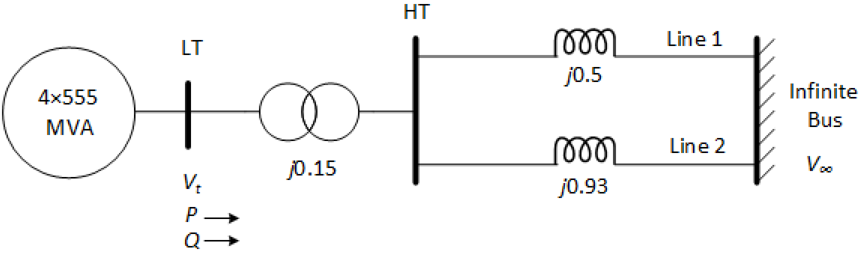
$$\Delta\dot{V}_{PSS} = \begin{bmatrix} -\frac{T_1}{T_2}\frac{\omega_{base}}{2H}K_2K_{PSS} & -\frac{T_1}{T_2}\frac{\omega_{base}}{2H}K_1K_{PSS} & -\frac{T_1}{T_2}\frac{\omega_{base}}{2H}DK_{PSS} & 0 & \left(\frac{1}{T_2} - \frac{1}{T_W}\frac{T_1}{T_2}\right) & -\frac{1}{T_2} \end{bmatrix} \begin{bmatrix} \Delta E_q' \\ \Delta\delta \\ \Delta\omega \\ \Delta E_{fd} \\ \Delta V_{WF} \\ \Delta V_{PSS} \end{bmatrix} + \begin{bmatrix} 0 & \frac{\omega_{base}}{2H}\frac{T_1}{T_2}K_{PSS} \end{bmatrix} \begin{bmatrix} \Delta V_{ref} \\ \Delta T_m \end{bmatrix} \tag{117}$$

The state space model including the PSS dynamics is

$$\begin{bmatrix} \Delta \dot{E}_q{}' \\ \Delta \dot{\delta} \\ \Delta \dot{\omega} \\ \Delta \dot{E}_{fd} \\ \Delta \dot{V}_{WF} \\ \Delta \dot{V}_{PSS} \end{bmatrix} = \begin{bmatrix} -\frac{1}{K_3 T'_{d0}} & -\frac{K_4}{T'_{d0}} & 0 & \frac{1}{T'_{d0}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{\omega_{base}}{2H} K_2 & -\frac{\omega_{base}}{2H} K_1 & -\frac{\omega_{base}}{2H} D & 0 & 0 & 0 \\ -\frac{K_A K_6}{T_A} & -\frac{K_A K_5}{T_A} & 0 & -\frac{1}{T_A} & 0 & \frac{K_A}{T_A} \\ -\frac{\omega_{base}}{2H} K_2 K_{PSS} & -\frac{\omega_{base}}{2H} K_1 K_{PSS} & -\frac{\omega_{base}}{2H} D K_{PSS} & 0 & -\frac{1}{T_W} & 0 \\ -\frac{T_1}{T_2}\frac{\omega_{base}}{2H} K_2 K_{PSS} & -\frac{T_1}{T_2}\frac{\omega_{base}}{2H} K_1 K_{PSS} & -\frac{T_1}{T_2}\frac{\omega_{base}}{2H} D K_{PSS} & 0 & \left(\frac{1}{T_2} - \frac{1}{T_W}\frac{T_1}{T_2}\right) & -\frac{1}{T_2} \end{bmatrix}$$

$$\begin{bmatrix} \Delta E'_q \\ \Delta \delta \\ \Delta \omega \\ \Delta E_{fd} \\ \Delta V_{WF} \\ \Delta V_{PSS} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{\omega_{base}}{2H} \\ \frac{K_A}{T_A} & 0 \\ 0 & \frac{\omega_{base}}{2H} K_{PSS} \\ 0 & \frac{\omega_{base}}{2H}\frac{T_1}{T_2} K_{PSS} \end{bmatrix} \begin{bmatrix} \Delta V_{ref} \\ \Delta T_m \end{bmatrix} \tag{118}$$

Example 12.4 from P. Kundur : This example analyzes the small signal stability of the example 12.3 including exciter dynamics with PSS.



The parameters of each of the four generators of the plant are in per unit on its rating are as follows:

$$X_d = 1.81 \quad X_q = 1.76 \quad X'_d = 0.3 \quad X_l = 0.16$$

$$R_s = 0.003 \quad T'_{do} = 8 \text{ s} \quad H = 3.5 \quad D = 0$$

The exciter parameters are

$$K_A = 200 \quad T_A = 0.02$$

The PSS parameters are

$$K_{PSS} = 9.5; \;\; T_W = 1.4; \;\; T_1 = 0.154; \;\; T_2 = 0.033$$

If the plant output is

$$P = 0.9 \quad Q = 0.3 \quad V_t = 1.0\angle 36^\circ \quad V_\infty = 0.995\angle 0^\circ$$

compute the following:

1. Eigenvalues of A.

For $K_A = 200$, the eigenvalues are

$$\lambda_1 = -46.4791$$

$$\lambda_2, \lambda_3 = -15.8405 \pm \jmath 14.6135$$

$$\lambda_4, \lambda_5 = -1.22 \pm +\jmath 6.6794$$

$$\lambda_6 = -0.741$$

Small Signal Stability Analysis of Multi-Machine System

- Let each generator be modelled by a sub-transient model along with a power system stabilizer and stator algebraic equations.
- The network is represented by real and reactive power balance equations.

Differential equations (for $i = 1, 2, \cdots, n_g$ generator):

$$T'_{d0i}\frac{dE'_{qi}}{dt} = -E'_{qi} - (X_{di} - X'_{di})[I_{di} - \frac{(X'_{di} - X''_{di})}{(X'_{di} - X_{lsi})^2}(-E'_{qi} + (X'_{di} - X_{lsi})I_{di} + \psi_{1di}] + E_{fdi} \quad (119)$$

$$T''_{d0i}\frac{d\psi_{1di}}{dt} = E'_{qi} - (X'_{di} - X_{lsi})I_{di} - \psi_{1di} \quad (120)$$

$$T'_{q0i}\frac{dE'_{di}}{dt} = -E'_{di} + (X_{qi} - X'_{qi})\{(I_{qi} - \frac{(X'_{qi} - X''_{qi})}{(X'_{qi} - X_{lsi})^2}(E'_{di} + (X'_{qi} - X_{lsi})I_{qi} + \psi_{2qi})\} \quad (121)$$

$$T''_{q0i}\frac{d\psi_{2qi}}{dt} = -E'_{di} - (X'_{qi} - X_{lsi})I_{qi} - \psi_{2qi} \quad (122)$$

$$\frac{d\delta_i}{dt} = \omega_i - \omega_s \tag{123}$$

$$\frac{2H_i}{\omega_{base}}\frac{d\omega_i}{dt} = T_{mi} - T_{ei} - D_i(\omega_i - \omega_{base}) \tag{124}$$

$$T_{Ei}\frac{dE_{fdi}}{dt} = -(K_{Ei} + S_{Ei}(E_{fdi}))E_{fdi} + V_{Ri} \tag{125}$$

$$T_A\frac{dV_R}{dt} = -V_R + K_A R_F - \frac{K_A K_F}{T_F}E_{fd} + K_{Ai}(V_{refi} - V_{ti}) \tag{126}$$

$$T_{Fi}\frac{dR_{Fi}}{dt} = -R_{Fi} + \frac{K_{Fi}}{T_{Fi}}E_{fdi} \tag{127}$$

$$\dot{V}_{WF} = -\frac{1}{T_{Wi}}V_{WFi} + K_{PSSi}\frac{\omega_{base}}{2H_i}(T_{mi} - T_{ei} - D_i(\omega_i - \omega_{base})) \tag{128}$$

$$\dot{V}_{PSS} = -\frac{1}{T_2} V_{PSSi} + \left(\frac{1}{T_{2i}} - \frac{1}{T_{Wi}} \frac{T_{1i}}{T_{2i}} \right) \Delta V_{WFi} + K_{PSSi} \frac{T_{1i}}{T_{2i}} \frac{\omega_{base}}{2H_i} (T_{mi} - T_{ei} - D_i(\omega_i - \omega_{base})) \tag{129}$$

$$T_{RHi} \frac{dT_{Mi}}{dt} = -\Delta T_{Mi} + \left(1 - \frac{K_{HPi} T_{RHi}}{T_{CHi}} \right) P_{CHi} + \frac{K_{HPi} T_{RHi}}{T_{CHi}} P_{SVi} \tag{130}$$

$$T_{CHi} \frac{dP_{CHi}}{dt} = -P_{CHi} + P_{SVi} \tag{131}$$

$$T_{SVi} \frac{dP_{SVi}}{dt} = -\frac{1}{R_{Di}} \left(\frac{\omega_i}{\omega_{base}} - 1 \right) + P_{refi} - P_{SVi}, \quad 0 \le P_{SVi} \le P_{SVi}^{max} \tag{132}$$

Stator algebraic equations :

$$V_i \sin(\delta_i - \theta_i) + R_{si} I_{di} - X''_{qi} I_{qi} - \frac{(X''_{qi} - X_{lsi})}{(X'_{qi} - X_{lsi})} E'_{di} + \frac{(X'_{qi} - X''_{qi})}{(X'_{qi} - X_{lsi})} \psi_{2qi} = 0 \tag{133}$$

$$V_i \cos(\delta_i - \theta_i) + R_{si} I_{qi} + X''_{di} I_{di} - \frac{(X''_{di} - X_{lsi})}{(X'_{di} - X_{lsi})} E'_{qi} - \frac{(X'_{di} - X''_{di})}{(X'_{di} - X_{lsi})} \psi_{1di} = 0 \tag{134}$$

Network Equations :
For $i = 1, 2, \cdots, n_g$ generator buses,

$$V_i \sin(\delta_i - \theta_i) I_{di} + V_i \cos(\delta_i - \theta_i) I qi - P_{Di}(V_i) - \sum_{j=1}^{n} V_i V_j Y_{ij} \cos(\alpha_{ij} + \theta_j - \theta_i) = 0 \tag{135}$$

$$V_i \cos(\delta_i - \theta_i) I_{di} - V_i \sin(\delta_i - \theta_i) I qi - Q_{Di}(V_i) + \sum_{j=1}^{n} V_i V_j Y_{ij} \sin(\alpha_{ij} + \theta_j - \theta_i) = 0 \tag{136}$$

For $i = n_g + 1, n_g + 2, \cdots, n$ load buses,

$$P_{Di}(V_i) + \sum_{j=1}^{n} V_i V_j Y_{ij} \cos(\alpha_{ij} + \theta_j - \theta_i) = 0 \tag{137}$$

$$Q_{Di}(V_i) - \sum_{j=1}^{n} V_i V_j Y_{ij} \sin(\alpha_{ij} + \theta_j - \theta_i) = 0 \tag{138}$$

The differential equations (119) to (132) can be represented as

$$\dot{X}_i = f_i(X_i, I_{di}, I_{qi}, \theta_g, V_g, U) \tag{139}$$

where

$$X_i = [E'_{qi} \ \psi_{1di} \ E'_{di} \ \psi_{2qi} \ \delta_i \ \omega_i \ V_{Ri} \ E_{fdi} \ R_{Fi} \ T_{Mi} \ P_{SVi} \ P_{CHi} \ V_{WFi} \ V_{PSSi}]^T$$
$$i = 1, 2, \cdots, n_g$$

$$U_i = \begin{bmatrix} V_{refi} & P_{refi} \end{bmatrix}^T \ i = 1, 2, \cdots, n_g$$

$$\theta_g = [\theta_1, \cdots, \theta_{n_g}]^T, \quad V_g = [V_1, \cdots, V_{n_g}]^T$$

$$\theta_l = [\theta_{n_g+1}, \cdots, \theta_n]^T, \quad V_l = [V_{n_g+1}, \cdots, V_n]^T$$

The stator equations (133) and (134) can be written as

$$g_i(X_i, I_{di}, I_{qi}, \theta_g, V_g) = 0 \; i = 1, \cdots, n_g \tag{140}$$

The network equations at the generator buses (135) and (136) can be written as

$$h_i(X_i, I_{di}, I_{qi}, \theta_g, V_g, \theta_l, V_l) = 0 \; i = 1, \cdots, n_g \tag{141}$$

The network equations at the load buses (137) and (138) can be written as

$$k_i(\theta_g, V_g, \theta_l, V_l) = 0 \; i = n_g + 1, \cdots, n \tag{142}$$

- The functions f, g, h, k are non linear.
- Let the steady state operating point be $[X_{i0}, I_{di0}, I_{qi0}, U_{i0}]$.
- The network voltages and angles at the steady state are $[\theta_{g0}, V_{g0}, \theta_{l0}, V_{l0}]$

Linearizing (139) around the initial operating conditions gives

$$\Delta \dot{X}_i = \underbrace{\left[\frac{\partial f_i}{\partial X_i}\right]}_{\mathrm{A_{1i}}} \Delta X_i + \underbrace{\left[\frac{\partial f_i}{\partial I_{di}} \quad \frac{\partial f_i}{\partial I_{qi}}\right]}_{\mathrm{B_{1i}}} \underbrace{\begin{bmatrix} \Delta I_{di} \\ \Delta I_{qi} \end{bmatrix}}_{\Delta \mathrm{I_{dqi}}} + \left[\underbrace{\frac{\partial f_i}{\partial \theta_g} \; \frac{\partial f_i}{\partial V_g}}_{\mathrm{B_{2i}}} \; 0 \; 0\right] \begin{bmatrix} \Delta \theta_g \\ \Delta V_g \\ \Delta \theta_l \\ \Delta V_l \end{bmatrix} + \underbrace{\left[\frac{\partial f_i}{\partial U_i}\right]}_{\mathrm{W_i}} \Delta U_i \tag{143}$$

$$\begin{aligned} \Delta \dot{X}_i = & \mathrm{A}_{1\mathrm{i}(14\times 14)} \Delta \mathrm{X}_{\mathrm{i}(14\times 1)} + \mathrm{B}_{1\mathrm{i}(14\times 2)} \Delta \mathrm{I}_{\mathrm{dqi}(2\times 1)} \\ & + [\mathrm{B}_{2\mathrm{i}(14\times 2n_g)} \; 0_{(14\times 2(n-n_g))}] \begin{bmatrix} \Delta \theta_g \\ \Delta V_g \\ \Delta \theta_l \\ \Delta V_l \end{bmatrix}_{(2n\times 1)} + \mathrm{W}_{\mathrm{i}(14\times 2)} \Delta \mathrm{U}_{\mathrm{i}(2\times 1)} \end{aligned} \tag{144}$$

For n_g generators,

$$\begin{aligned}
\Delta\dot{X} =& \mathrm{A}_{1\mathrm{i}(14n_g\times 14n_g)}\Delta\mathrm{X}_{(14n_g\times 1)} + \mathrm{B}_{1(14n_g\times 2)}\Delta\mathrm{I}_{\mathrm{dq}(2n_g\times 1)} \\
&+ [\mathrm{B}_{2(14n_g\times 2n_g)}\ 0_{(14n_g\times 2(n-n_g))}] \begin{bmatrix}\Delta\theta_g \\ \Delta V_g \\ \Delta\theta_l \\ \Delta V_l\end{bmatrix}_{(2n\times 1)} \\
&+ \mathrm{W}_{\mathrm{i}(14n_g\times 2n_g)}\Delta\mathrm{U}_{\mathrm{i}(2n_g\times 1)}
\end{aligned} \tag{145}$$

Similarly, by linearizing (140), we get

$$\underbrace{\left[\frac{\partial g_i}{\partial X_i}\right]}_{\mathrm{C_{1i}}}\Delta X_i + \underbrace{\left[\frac{\partial g_i}{\partial I_{di}} \quad \frac{\partial g_i}{\partial I_{qi}}\right]}_{\mathrm{C_{2i}}}\underbrace{\begin{bmatrix}\Delta I_{di}\\ \Delta I_{qi}\end{bmatrix}}_{\mathrm{\Delta I_{dqi}}} + \left[\underbrace{\frac{\partial g_i}{\partial \theta_g} \quad \frac{\partial g_i}{\partial V_g}}_{\mathrm{D_{1i}}} \quad 0 \; 0\right]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix} = 0 \tag{146}$$

$$\mathrm{C}_{1\mathrm{i}(2\times14)}\Delta\mathrm{X}_{\mathrm{i}(14\times1)} + \mathrm{C}_{2\mathrm{i}(2\times2)}\Delta\mathrm{I}_{\mathrm{dqi}(2\times1)} + [\mathrm{D}_{1\mathrm{i}(2\times2n_g)} \;\; 0_{(2\times2(n-n_g))}]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix}_{(2n\times1)} = 0 \tag{147}$$

For n_g generators,

$$\begin{aligned}\mathrm{C}_{1(2n_g\times14n_g)}\Delta\mathrm{X}_{(14n_g\times1)} &+ \mathrm{C}_{2(2n_g\times2n_g)}\Delta\mathrm{I}_{\mathrm{dq}(2n_g\times1)}\\ &+[\mathrm{D}_{1(2n_g\times2n_g)} \;\; 0_{(2n_g\times2(n-n_g))}]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix}_{(2n\times1)} = 0\end{aligned} \tag{148}$$

Similarly, by linearizing (141), we get

$$\underbrace{\left[\frac{\partial h_i}{\partial X_i}\right]}_{\mathrm{C_{3i}}}\Delta X_i + \underbrace{\left[\frac{\partial h_i}{\partial I_{di}} \quad \frac{\partial h_i}{\partial I_{qi}}\right]}_{\mathrm{C_{4i}}}\underbrace{\begin{bmatrix}\Delta I_{di}\\ \Delta I_{qi}\end{bmatrix}}_{\Delta \mathrm{I_{dqi}}} + \left[\underbrace{\frac{\partial h_i}{\partial \theta_g} \quad \frac{\partial h_i}{\partial V_g}}_{\mathrm{D_{2i}}} \quad \underbrace{\frac{\partial h_i}{\partial \theta_l} \quad \frac{\partial h_i}{\partial V_l}}_{\mathrm{D_{3i}}}\right]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix} = 0 \tag{149}$$

$$\mathrm{C_{3i}}_{(2\times 14)}\Delta \mathrm{X_i}_{(14\times 1)} + \mathrm{C_{4i}}_{(2\times 2)}\Delta \mathrm{I_{dqi}}_{(2\times 1)} + [\mathrm{D_{2i}}_{(2\times 2n_g)} \ \mathrm{D_{3i}}_{(2\times 2(n-n_g))}]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix}_{(2n\times 1)} = 0 \tag{150}$$

For n_g generators,

$$\begin{aligned}\mathrm{C_3}_{(2n_g\times 14n_g)}\Delta \mathrm{X}_{(14n_g\times 1)} + \mathrm{C_4}_{(2n_g\times 2n_g)}\Delta \mathrm{I_{dq}}_{(2n_g\times 1)}\\ + [\mathrm{D_2}_{(2n_g\times 2n_g)} \ \mathrm{D_3}_{(2n_g\times 2(n-n_g))}]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix}_{(2n\times 1)} = 0\end{aligned} \tag{151}$$

Similarly, by linearizing (142), we get

$$\begin{bmatrix} \underbrace{\frac{\partial k_i}{\partial \theta_g} \quad \frac{\partial k_i}{\partial V_g}}_{\text{D}_{4\text{i}}} & \underbrace{\frac{\partial k_i}{\partial \theta_l} \quad \frac{\partial k_i}{\partial V_l}}_{\text{D}_{5\text{i}}} \end{bmatrix} \begin{bmatrix} \Delta\theta_g \\ \Delta V_g \\ \Delta\theta_l \\ \Delta V_l \end{bmatrix} = 0 \tag{152}$$

$$[\text{D}_{4\text{i}(2\times 2n_g)} \ \text{D}_{5\text{i}(2\times 2(n-n_g))}] \begin{bmatrix} \Delta\theta_g \\ \Delta V_g \\ \Delta\theta_l \\ \Delta V_l \end{bmatrix}_{(2n\times 1)} = 0 \tag{153}$$

For n_g generators,

$$[\text{D}_{4(2n_g\times 2n_g)} \ \text{D}_{5(2n_g\times 2(n-n_g))}] \begin{bmatrix} \Delta\theta_g \\ \Delta V_g \\ \Delta\theta_l \\ \Delta V_l \end{bmatrix}_{(2n\times 1)} = 0 \tag{154}$$

Equations (145), (146), (151) and (154) can be written as

$$\Delta\dot{\mathsf{X}} = \mathsf{A}_1\Delta\mathsf{X} + \mathsf{B}_1\Delta\mathsf{I}_{\mathsf{dq}} + [\mathsf{B}_2\ 0]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix} + \mathsf{W}\Delta\mathsf{U} \tag{155}$$

$$0 = \mathsf{C}_1\Delta\mathsf{X} + \mathsf{C}_2\Delta\mathsf{I}_{\mathsf{dq}} + [\mathsf{D}_1\ 0]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix} \tag{156}$$

$$0 = \mathsf{C}_3\Delta\mathsf{X} + \mathsf{C}_4\Delta\mathsf{I}_{\mathsf{dq}} + [\mathsf{D}_2\ \mathsf{D}_3]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix} \tag{157}$$

$$0 = [\mathsf{D}_4\ \mathsf{D}_5]\begin{bmatrix}\Delta\theta_g\\ \Delta V_g\\ \Delta\theta_l\\ \Delta V_l\end{bmatrix} \tag{158}$$

From (158), the load bus voltages and angles can be determined as follows:

$$\begin{bmatrix} \Delta\theta_l \\ \Delta V_l \end{bmatrix} = -\mathsf{D_5}^{-1}\mathsf{D_4} \begin{bmatrix} \Delta\theta_g \\ \Delta V_g \end{bmatrix} \tag{159}$$

Let

$$\mathsf{D_6} = \mathsf{D_2} - \mathsf{D_3}\mathsf{D_5}^{-1}\mathsf{D_4} \tag{160}$$

Substituting (160) and (159) in (157) and (156), and finding the generator bus voltages, angles, direct axis current and quadrature axis current give

$$\begin{bmatrix} \Delta \mathsf{I_{dq}} \\ \Delta\theta_g \\ \Delta V_g \end{bmatrix} = -\begin{bmatrix} \mathsf{C_2} & \mathsf{D_1} \\ \mathsf{C_4} & \mathsf{D_6} \end{bmatrix}^{-1} \begin{bmatrix} \mathsf{C_1} \\ \mathsf{C_3} \end{bmatrix} \begin{bmatrix} \Delta \mathsf{X} \end{bmatrix} \tag{161}$$

By substituting (161) in (155),

$$\Delta\dot{\mathsf{X}} = \mathsf{A}_1\Delta\mathsf{X} + \begin{bmatrix}\mathsf{B}_1 & \mathsf{B}_2\end{bmatrix}\begin{bmatrix}\Delta\mathsf{I}_{\mathsf{dq}} \\ \Delta\theta_g \\ \Delta V_g\end{bmatrix} + \mathsf{W}\Delta\mathsf{U} \\ = \mathsf{A}_{sys}\Delta\mathsf{X} + \mathsf{W}\Delta\mathsf{U} \tag{162}$$

where

$$\mathsf{A}_{sys} = \mathsf{A}_1 - \begin{bmatrix}\mathsf{B}_1 & \mathsf{B}_2\end{bmatrix}\begin{bmatrix}\mathsf{C}_2 & \mathsf{D}_1 \\ \mathsf{C}_4 & \mathsf{D}_6\end{bmatrix}^{-1}\begin{bmatrix}\mathsf{C}_1 \\ \mathsf{C}_3\end{bmatrix}$$

- The small signal stability of the multi-machine system can now be known by finding the eigenvalues of the state transition matrix A_{sys}.
- Since, the rotor angle of any synchronous machine should be defined with respect to a reference, one state variable corresponding to the rotor angle of a generator which is taken as reference becomes redundant hence there will be one zero or a slightly positive eigenvalue corresponding to the redundancy of the reference angle.

Example 12.6 from P. Kundur :

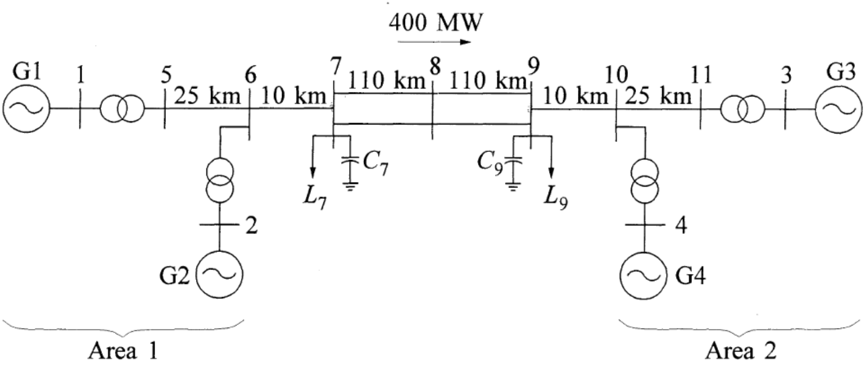


Figure: Two Area System

1. Perform Load flow.
2. Determine the machines' steady state values (δ, V_d, V_q, I_d, I_q, E'_q, E'_d, E_{fd}, ψ_{1d} and ψ_{2q}).
3. Calculate the system matrix as given in (162).
   - The size depends on the models.
   - For example, the sub-transient model of a synchronous machine without turbine, speed governor dynamics and PSS has seven states.

$$\Delta \mathrm{X}_i = [\Delta\delta_i \ \Delta\omega_i \ \Delta E'_{di} \ \Delta E'_{qi} \ \Delta\psi_{1di} \ \Delta\psi_{2qi} \ \Delta E_{fdi}]$$

   - Hence, the number of states is 28. The size of A is 28×28.
   - If PSS is considered, the number of states will increase. Suppose there is one lead-lag block, 2 additional states for each machine, hence the size of A will be 36×36
   - If there are two lead-lag blocks in PSS, 3 additional states for each machine,hence the size of A will be 40×40
4. Determine the eigenvalues of A.