Classical Model

- In this model, the stability of a single generator connected to an infinite bus is analyzed.
- Though this model is not suitable for current power systems, it helps us understand the basic phenomenon of stability.

Assumptions

1. Exciter dynamics are neglected and the filed current is assumed to be constant so that the induced voltage is always constant.
2. Damper winding dynamics are neglected.
3. The mechanical input power is assumed to be constant during the period of study.
4. Rotor is assumed to be of cylindrical type so that no saliency is present.

SMIB System

Let us consider Single Machine Infinite Bus (SMIB) system.

$$P_e = \frac{EV \sin\delta}{X}$$

Where $X = X_g + X_{Tr} + X_{TL}$ in p.u.

Rotor Dynamics - Swing Equation

The equation governing rotor motion of a synchronous machine is given as

$$J\frac{d^2\theta_m}{dt^2} = T_a = T_m - T_e \text{ N-m}$$

where

$J =$ the total moment of inertia of the rotor masses in kg-m^2
$\theta_m =$ the angular displacement of the rotor with respect to a stationary axis in mechanical radians (rad)
$t =$ time in seconds (s)
$T_m =$ the mechanical or shaft torque supplied by the prime mover in N-m
$T_e =$ the net electrical or electromagnetic torque in N-m
$T_a =$ the net accelerating torque in N-m

- T_m and T_e are considered positive for the synchronous generator.
- T_m accelerates the rotor in the positive θ_m in the direction of rotation.
- For a motor, T_m and T_e are reversed in sign.
- In the steady state, $T_m = T_e$. Hence, $T_a = 0$.

θ_m is measured with respect to a stationary reference axis on the stator. To represent it with respect to the synchronously rotating frame, let us define

$$\theta_m = \omega_{sm} t + \delta_m$$

where

ω_{sm} is the synchronous speed of the machine in mechanical radians per second

δ_m is the angular displacement of the rotor in mechanical radians from the synchronously rotating reference axis.

$$\frac{d\theta_m}{dt} = \omega_{sm} + \frac{d\delta_m}{dt}$$

$$\omega_m - \omega_{sm} = \frac{d\delta_m}{dt}$$

where $\omega_m = \dfrac{d\theta_m}{dt}$ is the angular velocity of the rotor in mechanical radians per second.

Differentiating it again,

$$\frac{d^2\theta_m}{dt^2} = \frac{d^2\delta_m}{dt^2}$$

Substituting it,

$$J\frac{d^2\delta_m}{dt^2} = T_a = T_m - T_e \text{ N-m}$$

On multiplying by ω_m,

$$J\omega_m\frac{d^2\delta_m}{dt^2} = P_a = P_m - P_e$$

where

P_m = shaft power input in MW

P_e = electrical power output in MW

P_a = accelerating power in MW

Let us define *inertia constant H*.

$$H = \frac{\text{stored kinetic energy in megajoules at synchronous speed}}{\text{Machine rating in MVA}}$$

$$H = \frac{\frac{1}{2} J \omega_{sm}^2}{S_{mach}} \text{ MJ/MVA}$$

Substituting it,

$$\frac{2H}{\omega_{sm}^2} \omega_m \frac{d^2 \delta_m}{dt^2} = \frac{P_a}{S_{mach}} = \frac{P_m - P_e}{S_{mach}}$$

In practice, ω_m does not differ significantly from the synchronous speed.

$\omega_m \approx \omega_{sm}$

$$\frac{2H}{\omega_{sm}} \frac{d^2 \delta_m}{dt^2} = P_a = P_m - P_e \text{ per unit}$$

It can be written as

$$\boxed{\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2} = P_m - P_e \text{ per unit}}$$

δ and ω_s have consistent units which can be mechanical or electrical degrees or radians.

- This equation is called the *swing equation* of the machine.
- It is a second-order nonlinear differential equation.
- When it is solved, we obtain δ as a function of t. This is called the *swing curve*.

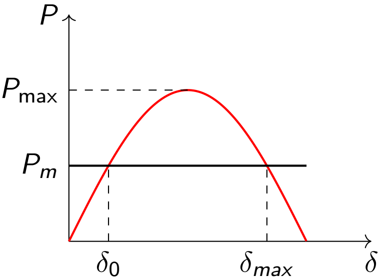
It can be written as two first-order differential equations.

$$\frac{2H}{\omega_s}\frac{d\omega}{dt} = P_m - P_e \text{ per unit}$$

$$\frac{d\delta}{dt} = \omega - \omega_s$$

- ω, ω_s and δ involve electrical radians or electrical degrees.
- δ is the **load angle**.

Small Signal Stability



Consider small incremental changes in δ around δ_0. The swing equation can be expressed as follows:

$$\frac{2H}{\omega_s}\frac{d^2(\delta_0+\Delta\delta)}{dt^2} = P_m - P_{max}\sin(\delta_0+\Delta\delta)$$

$$\frac{2H}{\omega_s}\frac{d^2\delta_0}{dt^2} + \frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} = P_m - P_{max}\sin\delta_0\cos\Delta\delta - P_{max}\cos\delta_0\sin\Delta\delta$$

Since $\Delta\delta$ is very small,

$$\sin\Delta\delta \approx \Delta\delta; \quad \cos\Delta\delta \approx 1$$

$$\frac{2H}{\omega_s}\frac{d^2\delta_0}{dt^2} + \frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} = P_m - P_{max}\sin\delta_0 - P_{max}\cos\delta_0\Delta\delta$$

At the stable equilibrium point (δ_0),

$$\frac{2H}{\omega_s}\frac{d^2\delta_0}{dt^2} = P_m - P_{max}\sin\delta_0 = 0$$

Therefore,

$$\frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} = -P_{max}\cos\delta_0\Delta\delta$$

Let $P_s = P_{max}\cos\delta_0$. P_s is called the synchronizing power or torque. In per unit system, torque and power are equal.

$$\frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} + P_s\Delta\delta = 0$$

- It is a linear second-order differential equation.
- When P_s is positive, the solution $\Delta\delta(t)$ is an undamped sinusoid.
- When P_s is negative, the solution $\Delta\delta(t)$ increases exponentially without limit.
- P_s is positive at δ_0 and negative at δ_{max}.
- Therefore δ_0 is a stable equilibrium point and δ_{max} is an unstable equilibrium point.

Let us apply the Laplace transformation to it.

$$\frac{2H}{\omega_s}s^2\Delta\delta(s) + P_s\Delta\delta(s) = 0$$

On solving, we get two roots.

$$s = \pm\sqrt{-\frac{P_s\omega_s}{2H}}$$

If P_s is positive,

$$s = \pm\jmath\sqrt{\frac{P_s\omega_s}{2H}}$$

The system has sustained oscillation. It is marginally stable.
If P_s is negative,

$$s = +\sqrt{\frac{P_s\omega_s}{2H}}, -\sqrt{\frac{P_s\omega_s}{2H}}$$

The system is unstable.

In synchronous machines, damper or amortisseur windings are there.

1. They damp out oscillations in generators.
2. They help synchronous motors start as induction motors because synchronous motors do not have starting torque.

The damping torque depends on the rate change of rotor angle.

$$P_d = D\frac{d\delta}{dt}$$

where D is the damping coefficient. After including this, the linearized swing equation can be written as

$$\frac{2H}{\omega_s}\frac{d^2\Delta\delta}{dt^2} + D\frac{d\Delta\delta}{dt} + P_s\Delta\delta = 0$$

Applying the Laplace transformation,

$$\frac{2H}{\omega_s}s^2\Delta\delta(s) + Ds\Delta\delta(s) + P_s\Delta\delta(s) = 0$$

$$s^2 + \frac{\omega_s}{2H}Ds + \frac{\omega_s}{2H}P_s = 0$$

By comparing this with the standard characteristics equation of a second order system,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

we get

$$\omega_n = \sqrt{\frac{\omega_s}{2H}P_s}$$

$$\zeta = \frac{D}{2}\sqrt{\frac{\omega_s}{2HP_s}}$$

where ω_n is the natural frequency of oscillations and ζ is the damping ratio.

The roots (eigen values) of the characteristics equation are

$$s = -\zeta\omega_n \pm \jmath\omega_n\sqrt{1-\zeta^2}$$

If P_s is positive

- there will be two complex conjugate roots with negative real part.
- hence the system will be stable.

If P_s is negative,

- there will be two real roots with one positive.
- hence the system will be unstable.

The linearized swing equations can also be written as follows:

$$\frac{2H}{\omega_s}\frac{d\Delta\omega}{dt} + D\Delta\omega + P_s\Delta\delta = 0$$
$$\frac{d\Delta\delta}{dt} = \Delta\omega$$

In state space form,

$$\begin{bmatrix} \Delta\dot{\omega} \\ \Delta\dot{\delta} \end{bmatrix} = \begin{bmatrix} -\frac{\omega_s D}{2H} & -\frac{\omega_s P_s}{2H} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega \\ \Delta\delta \end{bmatrix}$$

By taking the Laplace transformation,

$$\begin{bmatrix} s\Delta\omega(s) - \Delta\omega(0) \\ s\Delta\delta(s) - \Delta\delta(0) \end{bmatrix} = \begin{bmatrix} -\frac{\omega_s D}{2H} & -\frac{\omega_s P_s}{2H} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega(s) \\ \Delta\delta(s) \end{bmatrix}$$

$$\begin{bmatrix}\Delta\omega(s)\\ \Delta\delta(s)\end{bmatrix} = \begin{bmatrix} s + \dfrac{\omega_s D}{2H} & \dfrac{\omega_s P_s}{2H} \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix}\Delta\omega(0)\\ \Delta\delta(0)\end{bmatrix}$$

This can be written as

$$\begin{bmatrix}\Delta\omega(s)\\ \Delta\delta(s)\end{bmatrix} = \begin{bmatrix} s + 2\zeta\omega_n & \omega_n^2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix}\Delta\omega(0)\\ \Delta\delta(0)\end{bmatrix}$$

If the rotor angle is perturbed by a small angle, $\Delta\delta(0) = \Delta\delta$ and $\Delta\omega(0) = 0$.

$$\Delta\omega(s) = -\frac{\omega_n^2 \Delta\delta}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Delta\delta(s) = \frac{(s + 2\zeta\omega_n)\Delta\delta}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By taking the inverse Laplace transformation, we can get the following.

$$\Delta\omega(t) = -\frac{\omega_n \Delta\delta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t$$

and

$$\Delta\delta(t) = \frac{\Delta\delta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\theta = \cos^{-1}(\zeta)$.
The motion of rotor relative to the operating point is

$$\boxed{\delta = \delta_0 + \frac{\Delta\delta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta)}$$

and the rotor angular frequency is

$$\boxed{\omega = \omega_0 - \frac{\omega_n \Delta\delta}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t}$$

The time constant is

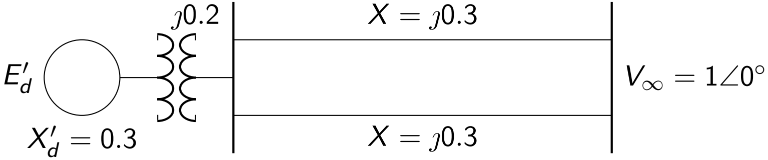
$$\tau = \frac{1}{\zeta\omega_n}$$

$$\tau = \frac{4H}{\omega_s D}$$

- The response approximately settles at 4 to 5 time constants.
- As H increase, it results in a longer settling time.
- As P_s increases, it results in an increase in ω_n.

Example

A 50 Hz synchronous generator having inertia constant $H = 5$ sec and a direct axis reactance $X'_d = 0.3$ p.u. is connected to an infinite bus through a transformer and a double circuit line. The network is purely reactive. The synchronous generator is delivering real power $P = 0.8$ p.u. and reactive power $Q = 0.074$ p.u. to the infinite bus of 1.0 p.u at steady state.



Assume the per unit damping power coefficient $D = 0.2$. Consider a small disturbance of $\Delta\delta = 10^\circ$. For example, the breakers open and quickly close. Determine the motion of rotor angle and the generator frequency.

The current flowing into the infinite bus is

$$I = \frac{S^*}{V^*} = \frac{0.8 - \jmath 0.074}{1\angle 0^\circ} = 0.8 - \jmath 0.074$$

The reactance between E'_d and V_∞ before the fault is

$$X = 0.3 + 0.2 + \frac{0.3 + 0.3}{2} = 0.65$$

The direct axis transient internal voltage is

$$E'_d = V_\infty + \jmath X_1 I = 1\angle 0^\circ + \jmath 0.65 \times (0.8 - \jmath 0.074) = 1.17\angle 26.38^\circ$$

$$P_{max} = \frac{E'_d V_\infty}{X_1} = \frac{1.17 \times 1}{0.65} = 1.8$$

$$\delta_0 = 26.38^\circ$$

$$P_s = P_{max}\cos\delta_0 = 1.8 \times cos(26.38^\circ) = 1.6125$$

$$\omega_n = \sqrt{\frac{\omega_s}{2H}P_s} = 7.1174 \text{ rad/s}$$

$$f_n = 1.133 \text{ Hz}$$

$$\zeta = \frac{D}{2}\sqrt{\frac{\omega_s}{2HP_s}} = 0.4414$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 6.3866 \text{ rad/s}$$

$$\theta = \cos^{-1}\zeta = 92.4^\circ$$

$$\delta = 26.38^\circ + 11.1444e^{-3.1416t}\sin(6.4t + 92.4^\circ)$$

$$f = 50 - 0.2525e^{-3.1416t}\sin 6.4t$$

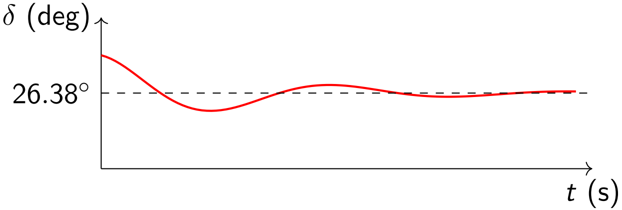


Figure: Swing Curve

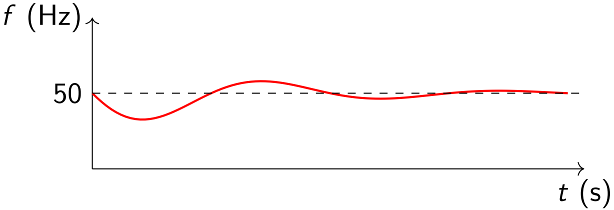


Figure: Frequency

Transient Stability

- The stability of a system is analyzed when it is subjected to large disturbances like faults.
- Since the disturbances are large, the rotor angle will swing high.
- This does not let us linearize the swing equation.
- The analysis of large disturbance stability is difficult.
- The equation has to be solved using numerical integration techniques.
- However, for an SMIB system, a direct approach called *Equal Area Criterion* method can be used to analyze it.
- It is a graphical approach.

Equal Area Criterion

$$\frac{2H}{\omega_s}\frac{d^2\delta}{dt^2} = P_m - P_e$$

$$\frac{d^2\delta}{dt^2} = \frac{\omega_s}{2H}(P_m - P_e)$$

Let us multiply both sides of the above equation by $2d\delta/dt$,

$$2\frac{d\delta}{dt}\frac{d^2\delta}{dt^2} = \frac{\omega_s(P_m - P_e)}{H}\frac{d\delta}{dt}$$

$$\frac{d}{dt}\left[\frac{d\delta}{dt}\right]^2 = \frac{\omega_s(P_m - P_e)}{H}\frac{d\delta}{dt}$$

On integration,

$$\left[\frac{d\delta}{dt}\right]^2 = \int_{\delta_0}^{\delta_m}\frac{\omega_s(P_m - P_e)}{H}d\delta$$

For a system to be stable, $\frac{d\delta}{dt} = 0$ after a disturbance.

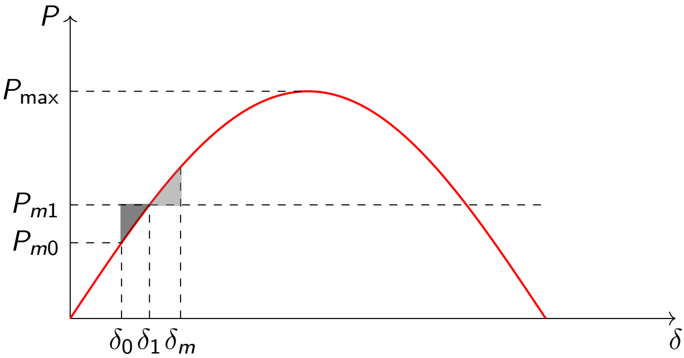
$$\int_{\delta_0}^{\delta_m} \frac{\omega_s(P_m - P_e)}{H} d\delta = 0$$

$$\int_{\delta_0}^{\delta_m} (P_m - P_e) d\delta = 0$$

where

$$\delta_0 = \text{initial rotor angle}$$
$$\delta_m = \text{maximum rotor angle}$$

Sudden change in P_m



$$\int_{\delta_0}^{\delta_m} (P_{m1} - P_e) d\delta = 0$$

δ_0 is the initial rotor angle. δ_m is the maximum rotor angle during oscillation.

$$\int_{\delta_0}^{\delta_1} (P_{m1} - P_e) d\delta + \int_{\delta_1}^{\delta_m} (P_{m1} - P_e) d\delta = 0$$

$$\int_{\delta_0}^{\delta_1} (P_{m1} - P_e) d\delta = \int_{\delta_1}^{\delta_m} (P_e - P_{m1}) d\delta$$

Therefore for the system to be stable

$$\text{Area}(A_1) = \text{Area}(A_2)$$

$$\text{Energy Gained } = \text{Energy Lost}$$

If $A_1 > A_2$, the system will be unstable.

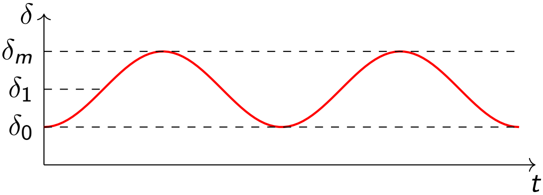
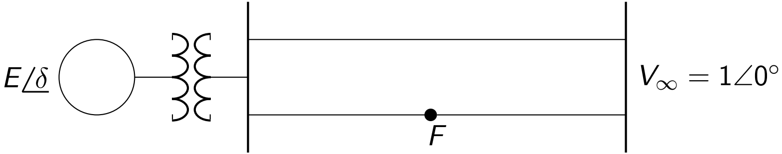


Figure: Swing Curve

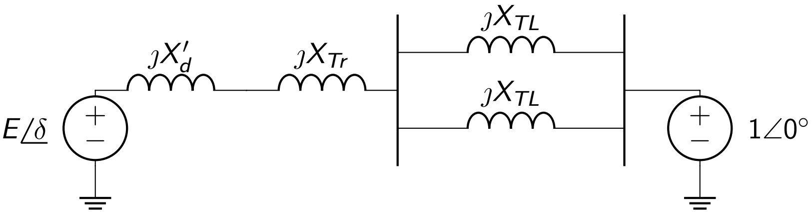
Damper or amortisseur windings damp out oscillations.

Short Circuit Faults



At point F, a three phase fault occurs. To analyze this, we need to understand the physical conditions before, during and after the fault.

1. Before Fault :



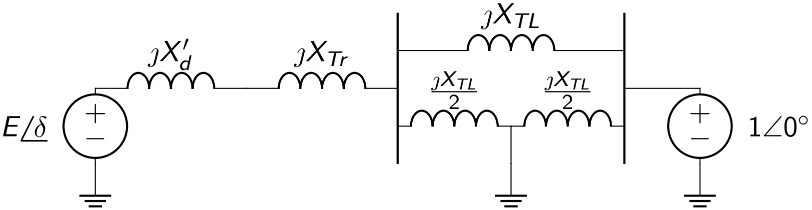
$$X_1 = X'_d + X_{Tr} + \frac{X_{TL}}{2}$$

$$P_{e1} = P_{max1} \sin\delta$$

where

$$P_{max1} = \frac{EV}{X_1}$$

2. During Fault :



- The total reactance X between two nodes can be found using $Y - \Delta$ conversion.
- X_2 during fault will be higher than before fault X_1.

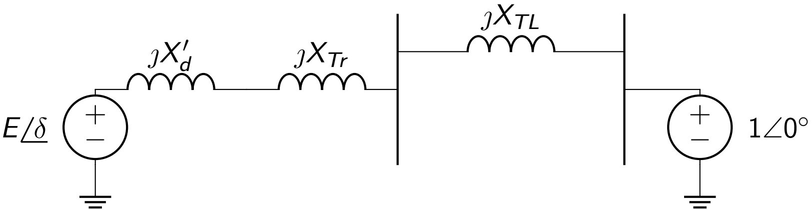
Hence,

$$P_{e2} = P_{max2} \sin \delta$$

where

$$P_{max2} = \frac{EV}{X_2}$$

3. After Fault :

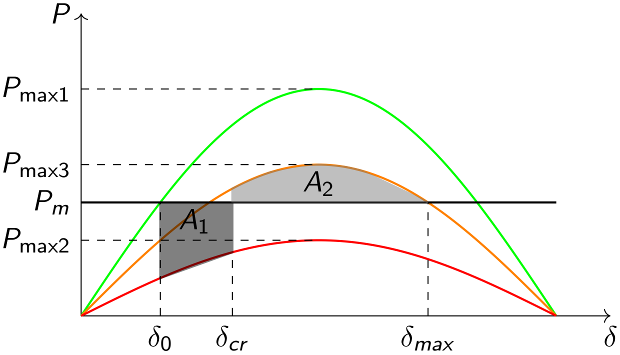


$$X_3 = X'_d + X_{Tr} + X_{TL}$$

$$P_{e3} = P_{max3} \sin\delta$$

where

$$P_{max3} = \frac{EV}{X_3}$$



$\delta_{max} = \pi - \sin^{-1}(\frac{P_m}{P_{max3}})$.
For the system to be stable, $A_1 = A_2$. There is a critical clearing angle δ_{cr} before which the fault has to be cleared.

$$\int_{\delta_0}^{\delta_{cr}} (P_m - P_{max2}\sin\delta)d\delta = \int_{\delta_{cr}}^{\delta_{max}} (P_{max3}\sin\delta - P_m)d\delta$$

Integrating and simplifying the above equation, we get

$$\cos\delta_{cr} = \frac{P_m(\delta_{max} - \delta_0) + P_{max3}\cos\delta_{max} - P_{max2}\cos\delta_0}{(P_{max3} - P_{max2})}$$

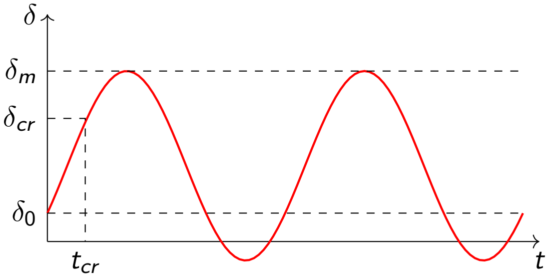
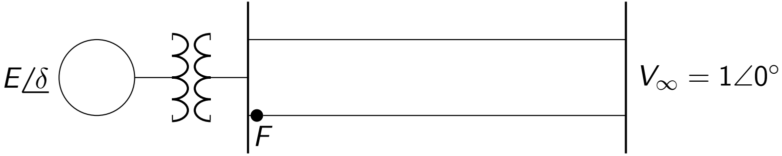


Figure: Swing Curve

t_{cr} is the critical clearing time in seconds.
It will settle at $\delta_{new} = \sin^{-1}(\frac{P_m}{P_{max3}})$ if damping is present.

Short Circuit Faults at the end of Transmission Lines (Near the bus)



1. Before Fault :

$$X_1 = X_d' + X_{Tr} + \frac{X_{TL}}{2}$$

$$P_{e1} = P_{max1} \sin \delta$$

where

$$P_{max1} = \frac{EV}{X_1}$$

2. During Fault :
   - Since the fault is near the bus, the bus voltage is zero.
   - The power transfer during fault is zero.

   Hence,

$$P_{e2} = 0$$

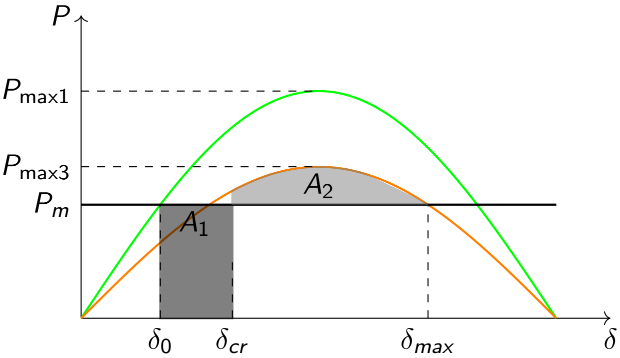
3. After Fault :

$$X_3 = X'_d + X_{Tr} + X_{TL}$$

$$P_{e3} = P_{max3} \sin \delta$$

   where

$$P_{max3} = \frac{EV}{X_3}$$



$\delta_{max} = \pi - \sin^{-1}(\frac{P_m}{P_{max3}})$.
For the system to be stable, $A_1 = A_2$.

$$\int_{\delta_0}^{\delta_{cr}} P_m d\delta = \int_{\delta_{cr}}^{\delta_{max}} (P_{max3}\sin\delta - P_m)d\delta$$

Integrating and simplifying the above equation, we get

$$\boxed{\cos\delta_{cr} = \frac{P_m(\delta_{max} - \delta_0) + P_{max3}\cos\delta_{max}}{P_{max3}}}$$

We can find the critical clearing time for this case as follows:

$$\frac{d^2\delta}{dt^2} = \frac{\omega_s}{2H}(P_m - P_e)$$

Since $P_e = 0$ during fault,

$$\frac{d^2\delta}{dt^2} = \frac{\omega_s}{2H}P_m$$

Integrating this,

$$\frac{d\delta}{dt} = \int_0^t \frac{\omega_s}{2H}P_m = \frac{\omega_s}{2H}P_m t$$

On further integration,

$$\delta = \frac{\omega_s}{4H}P_m t^2 + \delta_0$$

If $\delta = \delta_{cr}$,

$$\delta_{cr} = \frac{\omega_s}{4H} P_m t_{cr}^2 + \delta_0$$

$$t_{cr} = \sqrt{\frac{4H(\delta_{cr} - \delta_0)}{\omega_s P_m}}$$

Factors Influencing Transient Stability

1. How heavily the generator is loaded.
2. The generator output during the fault. This depends on the fault location and type.
3. The fault-clearing time.
4. The post fault transmission system reactance.
5. The generator inertia. The higher the inertia, the slower the rate of change in angle. This reduces A_1.
6. The generator internal voltage magnitude E. This depends on the field excitation.

Numerical Integration Methods

Let

$$\frac{d\mathrm{x}}{dt} = \mathrm{f}(\mathrm{x}, t)$$

where x is the state vector and $\mathrm{f}(\mathrm{x}, t)$ is a vector of non linear functions.

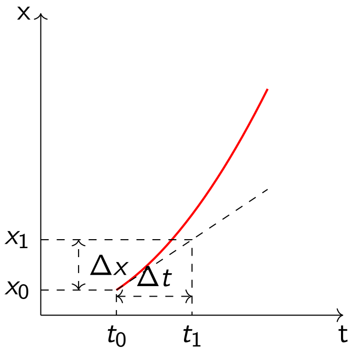
1. Explicit Methods
   1. Euler Method
   2. Modified Euler Method
   3. Runge-Kutta Methods
2. Implicit Methods
   1. Trapezoidal Rule

Euler Method

Consider the first-order differential equation.

$$\frac{dx}{dt} = f(x, t)$$

with $x = x_0$ at $t = t_0$.



At $x = x_0$, $t = t_0$, the curve can be approximated by its tangent having a slope

$$\left.\frac{dx}{dt}\right|_{x=x_0} = f(x_0, t_0)$$

Therefore,

$$\Delta x = \left.\frac{dx}{dt}\right|_{x=x_0} \Delta t$$

The value of x at $t = t_1 = t_0 + \Delta t$ is given by

$$x_1 = x_0 + \Delta x = x_0 + \left.\frac{dx}{dt}\right|_{x=x_0} \Delta t$$

This has to be repeated till the time reaches the final simulation time.

- Since it considers only the firs derivative of x, it is referred to as a *first order method*.
- Δt has to be small to achieve accuracy.
- Since it uses only the first order information, it may introduce error.

Modified Euler Method

- The standard Euler method results in inaccuracies because it uses the derivative at the beginning of the interval.
- The modified Euler method tries to overcome this issue by using the average of the derivatives at the two ends.

It consists of the following steps.

1. Predictor step

$$x_1^p = x_0 + \left.\frac{dx}{dt}\right|_{x=x_0} \Delta t$$

2. Corrector step

$$x_1^c = x_0 + \frac{1}{2}\left(\left.\frac{dx}{dt}\right|_{x=x_0} + \left.\frac{dx}{dt}\right|_{x=x_1^p} \right) \Delta t$$

This process has to be repeated until the desired accuracy or the final simulation time.

Runge-Kutta (R-K) Methods

- Euler and the modified Euler method require smaller time steps.
- R-K methods approximate the Taylor series solution. However they do not need derivatives higher than the first.
- R-K methods use the effectiveness of higher derivatives by several evaluations of the first derivative.
- They are classified based on the number of evaluations.

Second order R- K Method

The value of x at $t = t_0 + \Delta t$ is

$$x_1 = x_0 + \Delta x = x_0 + \frac{k_1 + k_2}{2}$$

where

$$k_1 = f(x_0, t_0)\Delta t$$

$$k_2 = f(x_0 + k_1, t_0 + \Delta t)\Delta t$$

In general,

$$x_{n+1} = x_n + \frac{k_1 + k_2}{2}$$

where

$$k_1 = f(x_n, t_n)\Delta t$$

$$k_2 = f(x_n + k_1, t_n + \Delta t)\Delta t$$

Fourth order R- K Method

The value x at $n+1^{st}$ step is

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = f(x_n, t_n)\Delta t$$

$$k_2 = f(x_n + \frac{k_1}{2}, t_n + \frac{\Delta t}{2})\Delta t$$

$$k_3 = f(x_n + \frac{k_2}{2}, t_n + \frac{\Delta t}{2})\Delta t$$

$$k_4 = f(x_n + k_3, t_n + \Delta t)\Delta t$$

$$
\begin{aligned}
k_1 &= (\text{slope at the beginning of time step})\Delta t \\
k_2 &= (\text{first approximation to slope at mid step})\Delta t \\
k_3 &= (\text{second approximation to slope at mid step})\Delta t \\
k_4 &= (\text{slope at the end of step})\Delta t \\
\Delta x &= \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

This is equivalent to considering upto fourth derivative terms in the Taylor series expansion.

Stability of Explicit Methods

Explicit methods calculate x at any time step from the knowledge of the values of x at previous time steps.

- They are not numerically stable.
- They require smaller time steps.
- For the stiff systems (Stiffness is the ratio of the largest to smallest time constants), they blow up unless a small time step is used.
- Stiffness can also be found by the ratio of the largest to smallest eigenvalues of the linearized system.

Implicit Methods

Consider the differential equation

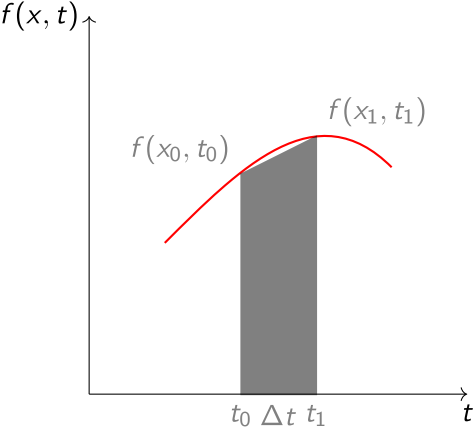
$$\frac{dx}{dt} = f(x, t)$$

with $x = x_0$ and $t = t_0$.

$$x_1 = x_0 + \int_{t_0}^{t_1} f(x, \tau) d\tau$$

- Implicit methods approximate the integral.
- The simplest implicit integration method is the *trapezoidal rule*.
- The trapezoidal rule approximates the integral by trapezoids.

Trapezoidal Rule



The trapezoidal rule is

$$x_1 = x_0 + \frac{\Delta t}{2}[f(x_0, t_0) + f(x_1, t_1)]$$

In general

$$x_{n+1} = x_n + \frac{\Delta t}{2}[f(x_n, t_n) + f(x_{n+1}, t_{n+1})]$$

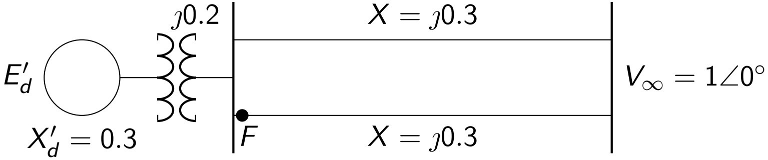
- x_{n+1} appears on both sides.
- Implicit methods find x using x at the previous time step as well as current value.
- Since the current value is unknown, an implicit equation must be solved.

Stability of Implicit Methods

- Implicit methods are numerically stable
- For the stiff systems, implicit methods suffer from accuracy but not numerical stability.
- Implicit methods work well with larger time steps.
- For systems where time steps are limited by numerical stability rather than accuracy, implicit methods are better.

Example

A 50 Hz synchronous generator having inertia constant $H = 5$ sec and a direct axis reactance $X'_d = 0.3$ p.u. is connected to an infinite bus through a transformer and a double circuit line. The network is purely reactive. The synchronous generator is delivering real power $P = 0.8$ p.u. and reactive power $Q = 0.074$ p.u. to the infinite bus of 1.0 p.u at steady state.



A solid three phase fault occurs at point F and the fault is cleared by opening the faulted line.

1. Determine the critical clearing angle and the critical fault clearing time using numerical integration.
2. Check the above angle using the Equal Area Criterion.

The current flowing into the infinite bus is

$$I = \frac{S^*}{V^*} = \frac{0.8 - \jmath 0.074}{1\angle 0^\circ} = 0.8 - \jmath 0.074$$

The reactance between E'_d and V_∞ before the fault is

$$X_1 = 0.3 + 0.2 + \frac{0.3 + 0.3}{2} = 0.65$$

The direct axis transient internal voltage is

$$E'_d = V_\infty + \jmath X_1 I = 1\angle 0^\circ + \jmath 0.65 \times (0.8 - \jmath 0.074) = 1.17\angle 26.38^\circ$$

$$P_{max1} = \frac{E'_d V_\infty}{X_1} = \frac{1.17 \times 1}{0.65} = 1.8$$

$$\delta_0 = 26.38^\circ$$

The reactance during the fault is

$$X_2 = \infty$$

$$P_{max2} = 0$$

The reactance after the fault is

$$X_3 = 0.3 + 0.2 + 0.3 = 0.8$$

$$P_{max3} = \frac{1.17 \times 1}{0.8} = 1.46$$

The swing equation can be written as follows:

$$\frac{d\omega}{dt} = \frac{\omega_s}{2H}(P_m - P_e)$$
$$\frac{d\delta}{dt} = (\omega - \omega_s)$$

The general formula for the second order R-K method

$$\omega_{n+1} = \omega_n + \left(\frac{k_{1\omega} + k_{2\omega}}{2}\right)$$
$$\delta_{n+1} = \delta_n + \left(\frac{k_{1\delta} + k_{2\delta}}{2}\right)$$

where

$$k_{1\omega} = \frac{\omega_s}{2H}(P_m - P_{max}\sin(\delta_n))\Delta t$$
$$k_{1\delta} = (\omega_n - \omega_s)\Delta t$$
$$k_{2\omega} = \frac{\omega_s}{2H}(P_m - P_{max}\sin(\delta_n + k_{1\omega}))\Delta t$$
$$k_{2\delta} = (\omega_n + k_{1\delta} - \omega_s)\Delta t$$

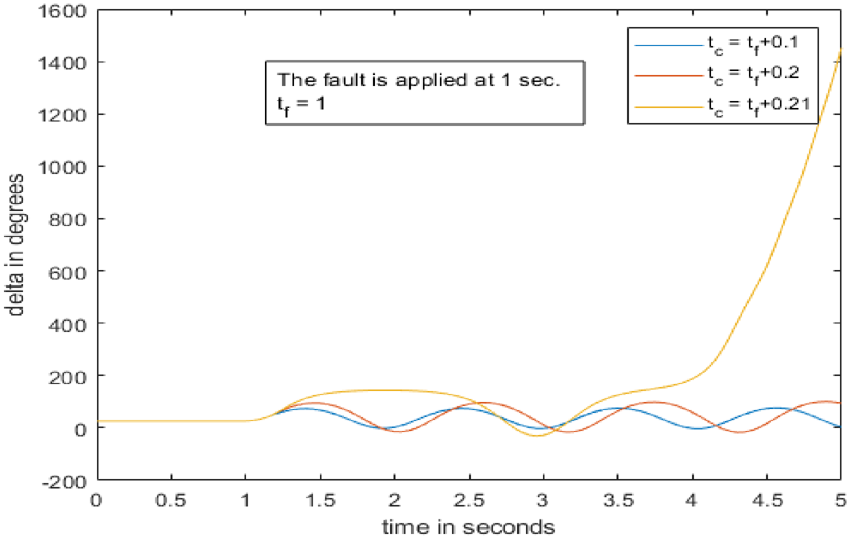


Figure: Swing Curve

- The fault was applied at $t = 1$ sec.
- $\Delta t = 0.05$ sec.
- The simulation was carried till $t_f = 5$ sec.
- It is found that the critical clearing time $t_{cr} = 0.2$ sec.

$$\delta_{cr} = \frac{\omega_s}{4H} P_m t_{cr}^2 + \delta_0$$

$$\delta_{cr} \approx 55^\circ$$

By using the equal area criterion

$$\boxed{\cos\delta_{cr} = \frac{P_m(\delta_{max} - \delta_0) + P_{max3}\cos\delta_{max}}{P_{max3}}}$$

$$P_{max3} = 1.4625$$

$$\delta_{max} = \pi - \sin^{-1}\left(\frac{P_m}{P_{max3}}\right) = 2.3887$$

$$\delta_{cr} \approx 54^\circ$$

Drawbacks of the classical model

1. The dynamics of rotor field winding and the damper winding on the generator are totally neglected in the classical model. However, they can effect the stability of a system significantly.
2. In the classical model, the internal voltage behind the transient reactance was assumed to be constant. This is not true since the rotor filed current is controlled through an exciter and automatic voltage regulator (AVR). Their dynamics have to be included.
3. In the classical model, P_m is assumed to be constant. But P_m depends on speed governor and turbine dynamics. Their dynamics need to be considered.
4. Dynamic loads like induction motors, synchronous motors, power electronic devices do affect the stability. In the classical model, they were not taken into consideration.