Network Matrices

- Per unit impedance and reactance diagrams are required to analyze power systems.
- For interconnected systems, network matrices have to be formed to analyze them.
- There are two matrices.
  1. bus admittance matrix Y_{bus}.
  2. bus impedance matrix Z_{bus}.
- Y_{bus} is mainly used in load flow studies. Whereas Z_{bus} is mainly used in short circuit studies.
- These two matrices are related by

$$Z_{bus} = Y_{bus}^{-1}$$

Bus Admittance Matrix (Y_{bus}) :

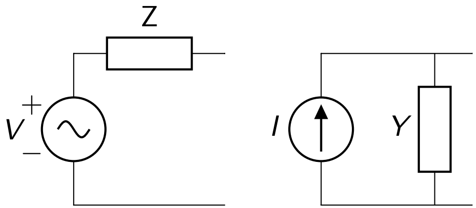
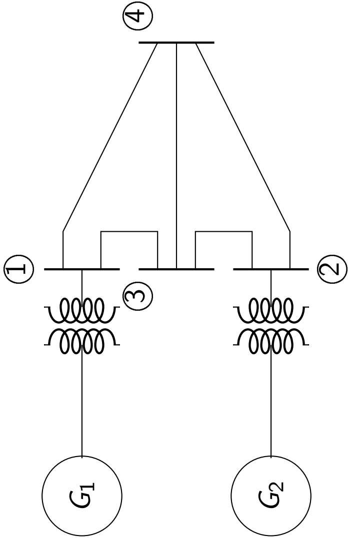


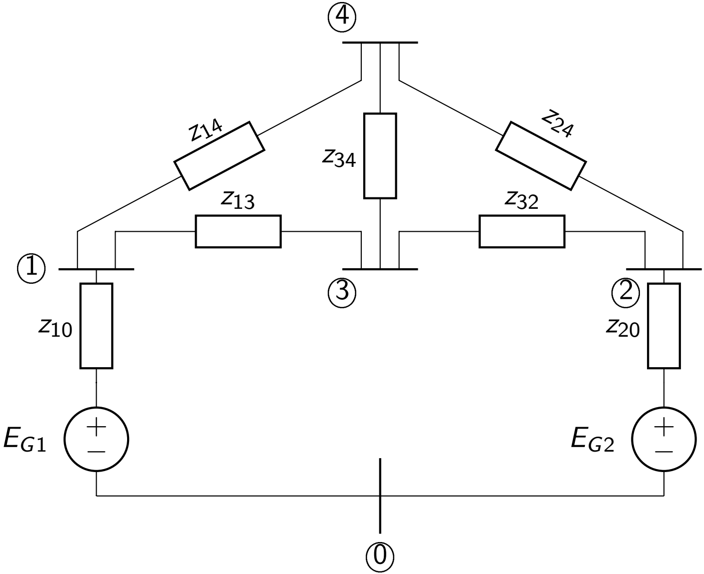
Figure: Thevenin - Norton Equivalent

$$I = \frac{V}{Z} \quad Y = \frac{1}{Z}$$

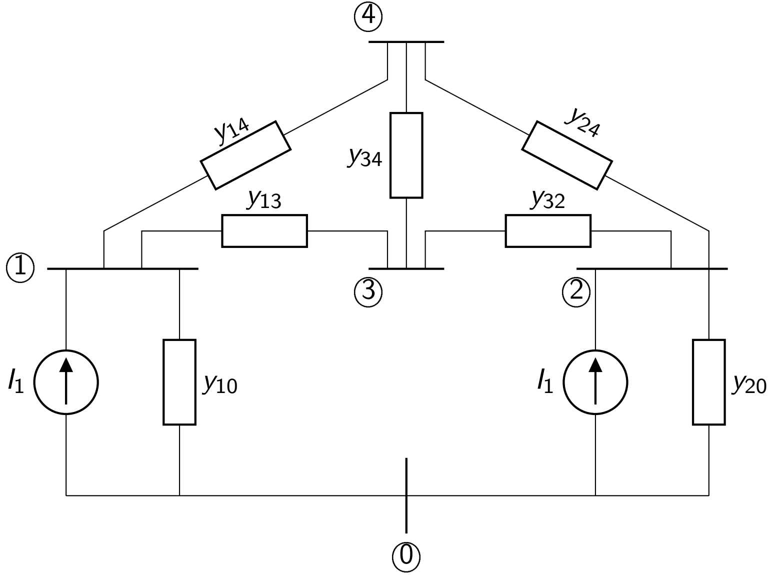
Let us consider a four bus system.



It is assumed that the short line model is used.



By using Thevenin - Norton equivalent and converting all impedances to admittances,



By applying KCL at node ①,

$$I_1 = y_{10}V_1 + y_{13}(V_1 - V_3) + y_{14}(V_1 - V_4)$$

$$I_1 = (y_{10} + y_{13} + y_{14})V_1 - y_{13}V_3 - y_{14}V_4$$

At node ②,

$$I_2 = y_{20}V_2 + y_{23}(V_2 - V_3) + y_{24}(V_2 - V_4)$$

$$I_2 = (y_{20} + y_{23} + y_{24})V_2 - y_{23}V_3 - y_{24}V_4$$

At node ③,

$$0 = y_{13}(V_3 - V_1) + y_{34}(V_3 - V_4) + y_{32}(V_3 - V_2)$$

$$0 = -y_{13}V_1 - y_{32}V_2 + (y_{13} + y_{32} + y_{34})V_3 - y_{34}V_4$$

At node ④,

$$0 = y_{14}(V_4 - V_1) + y_{24}(V_4 - V_2) + y_{34}(V_4 - V_3)$$

$$0 = -y_{14}V_1 - y_{24}V_2 - y34V_3 + (y_{14} + y_{24} + y_{34})V_4$$

By arranging the equations in a Matrix form,

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (y_{10}+y_{13}+y_{14}) & 0 & -y_{13} & -y_{14} \\ 0 & (y_{20}+y_{23}+y_{24}) & -y_{23} & -y_{24} \\ -y_{13} & -y_{32} & (y_{13}+y_{32}+y_{34}) & -y_{34} \\ -y_{14} & -y_{24} & -y_{34} & (y_{14}+y_{24}+y_{34}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

In general, for a 4 node (bus) system

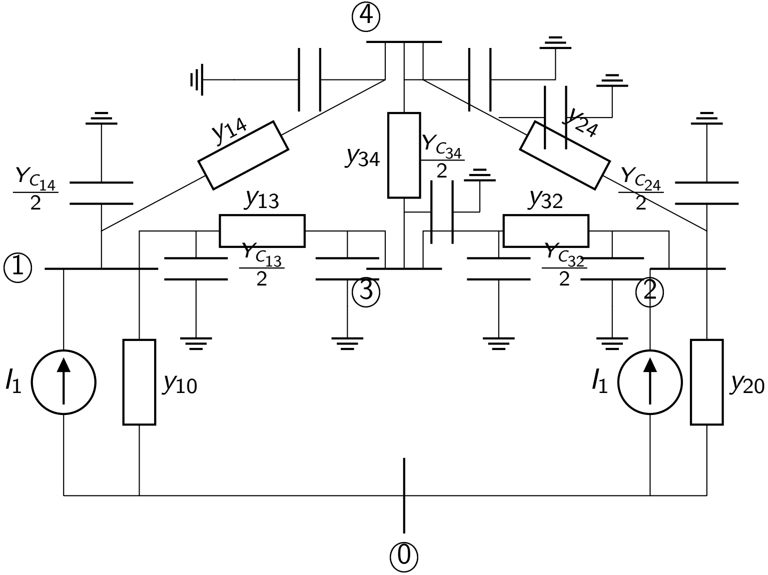
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

where

$$Y_{11} = (y_{10}+y_{13}+y_{14})$$
$$Y_{12} = Y_{21} = 0 \qquad Y_{22} = (y_{20}+y_{23}+y_{24})$$
$$Y_{13} = Y_{31} = -y_{13} \qquad Y_{23} = Y_{32} = -y_{32} \qquad Y_{33} = (y_{13}+y_{32}+y_{34})$$
$$Y_{14} = Y_{41} = -y_{14} \qquad Y_{24} = Y_{42} = -y_{24} \qquad Y_{34} = Y_{43} = -y_{34} \qquad Y_{44} = (y_{14}+y_{24}+y_{34})$$

Inclusion of Line Charging Capacitors

Transmission lines are modeled using π model.



By applying KCL at all the nodes and arranging the equations in a Matrix form,

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}$$

where

$$Y_{11} = (y_{10} + y_{13} + y_{14}) + (\frac{Y_{C_{13}}}{2} + \frac{Y_{C_{14}}}{2})$$

$$Y_{12} = Y_{21} = 0$$

$$Y_{13} = Y_{31} = -y_{13}$$

$$Y_{14} = Y_{41} = -y_{14}$$

$$Y_{22} = (y_{20} + y_{23} + y_{24}) + (\frac{Y_{C_{32}}}{2} + \frac{Y_{C_{24}}}{2})$$

$$Y_{23} = Y_{32} = -y_{32}$$

$$Y_{24} = Y_{42} = -y_{24}$$

$$Y_{33} = (y_{13} + y_{32} + y_{34}) + (\frac{Y_{C_{32}}}{2} + \frac{Y_{C_{13}}}{2} + \frac{Y_{C_{34}}}{2})$$

$$Y_{34} = Y_{43} = -y_{34}$$

$$Y_{44} = (y_{14} + y_{24} + y_{34}) + (\frac{Y_{C_{14}}}{2} + \frac{Y_{C_{24}}}{2} + \frac{Y_{C_{34}}}{2})$$

For an n bus system,

$$\mathrm{I}_{bus} = \mathrm{Y}_{bus}\mathrm{V}_{bus}$$

where

I_{bus} is a vector of **injected currents** in to the network ($n \times 1$).

Y_{bus} is a bus admittance matrix ($n \times n$).

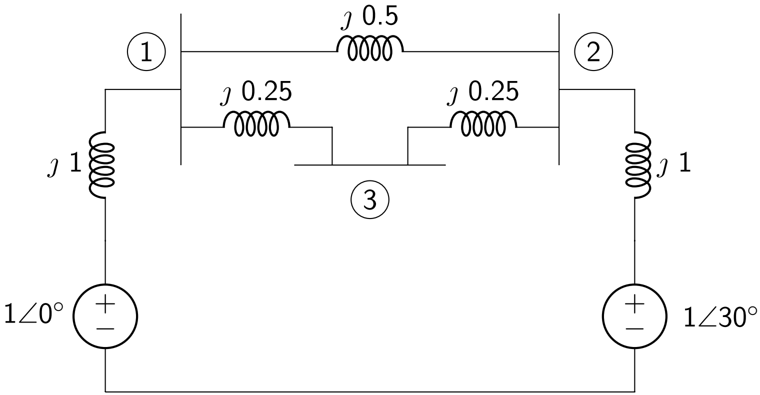
V_{bus} is a vector of bus voltages ($n \times 1$).

- Y_{bus} is symmetrical. (It will be unsymmetrical when a phase shifting transformer is used in a system.)
- Y_{bus} can be formed by inspection.
- Y_{bus} is generally sparse (many zeroes).

Two simple rules to form Y_{bus} by inspection:

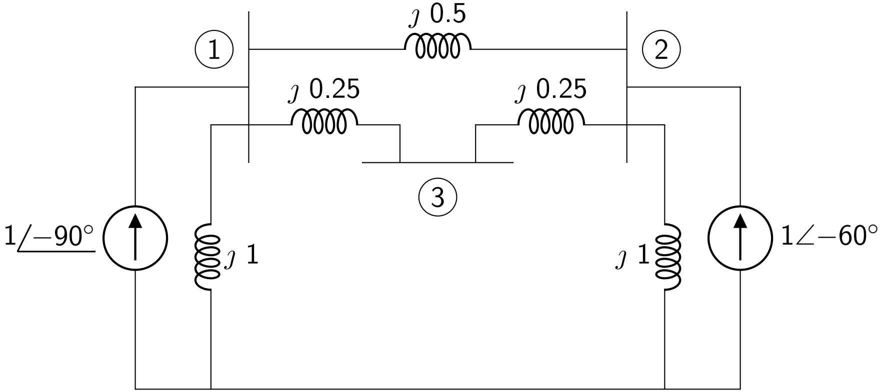
1. Diagonal Elements:
   Y_{ii} = sum of the admittances directly connected to node ⓘ.
2. Off-diagonal elements:
   $\mathrm{Y}_{ij} = \mathrm{Y}_{ji}$ = *negative* of the net admittance connected between nodes ⓘ and ⓙ.

Example 1: Consider the reactance network.



1. Form Y_{bus} matrix.
2. Find the voltages at each bus.

By Thevenin - Norton transformation,



1.

$$Y_{bus} = \begin{bmatrix} (\frac{1}{j1} + \frac{1}{j0.25} + \frac{1}{j0.5}) & -\frac{1}{j0.5} & -\frac{1}{j0.25} \\ -\frac{1}{j0.5} & (\frac{1}{j1} + \frac{1}{j0.25} + \frac{1}{j0.5}) & -\frac{1}{j0.25} \\ -\frac{1}{j0.25} & -\frac{1}{j0.25} & (\frac{1}{j0.25} + \frac{1}{j0.25}) \end{bmatrix}$$

$$Y_{bus} = \begin{bmatrix} -j7 & j2 & j4 \\ j2 & -j7 & j4 \\ j4 & j4 & -j8 \end{bmatrix}$$

2.

$$\mathrm{Y}_{bus}\mathrm{V}_{bus} = \mathrm{I}_{bus}$$

$$\begin{bmatrix} -\jmath 7 & \jmath 2 & \jmath 4 \\ \jmath 2 & -\jmath 7 & \jmath 4 \\ \jmath 4 & \jmath 4 & -\jmath 8 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1\underline{/-90^\circ} \\ 1\angle -60^\circ \\ 0 \end{bmatrix}$$

By solving this,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0.9663\underline{/13.3^\circ} \\ 0.9664\underline{/16.7^\circ} \\ 0.966\underline{/15^\circ} \end{bmatrix}$$

If the size of the matrix is more than 3, we need to use Gaussian Elimination to solve it.

Node Elimination - Kron Reduction

Let us consider a three node network.

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

Suppose there is no injection at node ③. $I_3 = 0$.

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix}$$

Since there is no injection (neither generation nor load) at ③, this can be eliminated.

$$Y_{11} V_1 + Y_{12} V_2 + Y_{13} V_3 = I_1$$
$$Y_{21} V_1 + Y_{22} V_2 + Y_{23} V_3 = I_2$$
$$Y_{31} V_1 + Y_{32} V_2 + Y_{33} V_3 = 0$$

From the last equation,

$$V_3 = -\frac{1}{Y_{33}}(Y_{31}V_1 + Y_{32}V_2)$$

Substituting V_3 in the first two equations, we get

$$(Y_{11} - \frac{Y_{13}Y_{31}}{Y_{33}})V_1 + (Y_{12} - \frac{Y_{13}Y_{32}}{Y_{33}})V_2 = I_1$$
$$(Y_{21} - \frac{Y_{23}Y_{31}}{Y_{33}})V_1 + (Y_{22} - \frac{Y_{23}Y_{32}}{Y_{33}})V_2 = I_2$$

Arranging the above equations in a matrix form,

$$\begin{bmatrix} (Y_{11} - \frac{Y_{13}Y_{31}}{Y_{33}}) & (Y_{12} - \frac{Y_{13}Y_{32}}{Y_{33}}) \\ (Y_{21} - \frac{Y_{23}Y_{31}}{Y_{33}}) & (Y_{22} - \frac{Y_{23}Y_{32}}{Y_{33}}) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

The new Y_{bus} is

$$\text{Y}_{bus(new)} = \begin{bmatrix} (Y_{11} - \frac{Y_{13}Y_{31}}{Y_{33}}) & (Y_{12} - \frac{Y_{13}Y_{32}}{Y_{33}}) \\ (Y_{21} - \frac{Y_{23}Y_{31}}{Y_{33}}) & (Y_{22} - \frac{Y_{23}Y_{32}}{Y_{33}}) \end{bmatrix}$$

The size is now reduced by 1.
Let us generalize the above procedure. Suppose the node ⓚ is to be eliminated from n nodes.

$$Y_{ij(new)} = Y_{ij} - \frac{Y_{ik}Y_{kj}}{Y_{kk}} \quad i,j = 1,2\cdots n \quad i,j \neq k$$

The size of the new Y_{bus} matrix is $(n-1) \times (n-1)$.

Example 2 : Let us consider the example 1.

$$\mathrm{Y}_{bus} = \begin{bmatrix} -\jmath 7 & \jmath 2 & \jmath 4 \\ \jmath 2 & -\jmath 7 & \jmath 4 \\ \jmath 4 & \jmath 4 & -\jmath 8 \end{bmatrix}$$

Let us eliminate the node ③.

$$Y_{11(new)} = Y_{11} - \frac{Y_{13}Y_{31}}{Y_{33}} = -\jmath 7 - \frac{-16}{-\jmath 8} = -\jmath 5$$

$$Y_{12(new)} = \jmath 2 - \frac{-16}{-\jmath 8} = \jmath 4$$

$$Y_{21(new)} = Y_{12(new)} = \jmath 4$$

$$Y_{22(new)} = -\jmath 7 - \frac{-16}{-\jmath 8} = -\jmath 5$$

$$\mathrm{Y}_{bus(new)} = \begin{bmatrix} -\jmath 5 & \jmath 4 \\ \jmath 4 & -\jmath 5 \end{bmatrix}$$

Bus Impedance Matrix Z_{bus} :

$$Z_{bus} = Y_{bus}^{-1}$$

- If Y_{bus} is symmetric, Z_{bus} must also be symmetric.
- Unlike Y_{bus}, Z_{bus} is a full matrix.
- Z_{bus} is used in fault calculations.

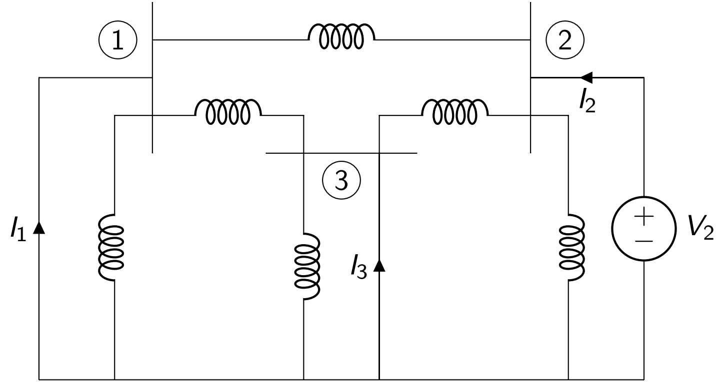
For a network of 3 nodes,

$$Z_{bus} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix}$$

- The diagonal elements are called the driving point impedances of the buses.
- The off-diagonal elements are called the transfer impedances of the buses.

Let us compare the elements of Z_{bus} with Y_{bus}.

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$



The self admittance Y_{22} can be found as follows:

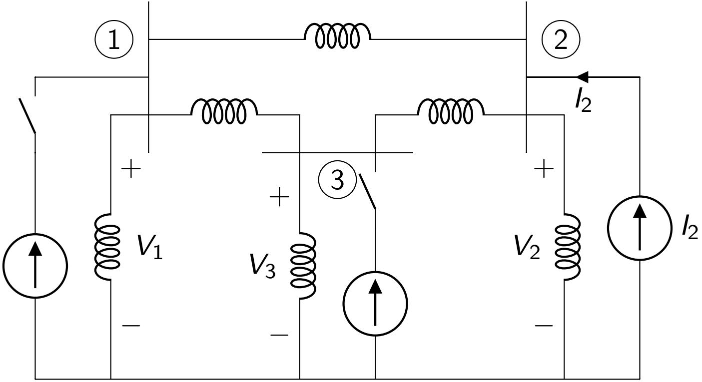
$$Y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=V_3=0}$$

The mutual admittance Y_{12} can be found as follows:

$$Y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=V_3=0}$$

Let us find the elements of Z_{bus}.

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$



The driving point impedance Z_{22} is determined as follows:

$$Z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=I_3=0}$$

$$Z_{22} \neq \frac{1}{Y_{22}}$$

The transfer impedance Z_{12} is found as follows:

$$Z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=I_3=0}$$

$$Z_{12} \neq \frac{1}{Y_{12}}$$

- Z_{22} is the Thevenin Impedance at node ②.
- Similarly, all the diagonal elements of Z_{bus} are the Thevenin impedance at the respective buses.

Modification of an Existing Z_{bus} :

- An existing Z_{bus} is modified whenever a new bus or line is added to the network.
- If we know how to modify an existing Z_{bus}, we can *build* it directly.

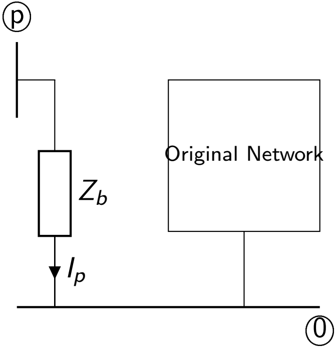
There are normally four cases

1. Adding a new bus to the reference bus through an impedance
2. Adding a new bus to an existing bus through an impedance
3. Adding an impedance between an existing bus and the reference bus
4. Adding an impedance between two existing buses

Let us assume that there are n buses in the existing network.

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{n1} & \cdots & Z_{nn} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix}$$

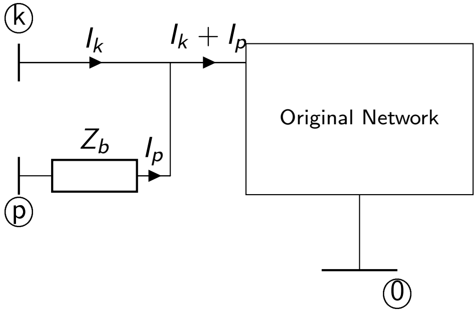
1. Adding a new bus to the reference bus through an impedance



A new bus p is added to the network through an impedance Z_b.

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \\ V_p \end{bmatrix} = \underbrace{\begin{bmatrix} Z_{11} & \cdots & Z_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ Z_{n1} & \cdots & Z_{nn} & 0 \\ 0 & \cdots & 0 & Z_b \end{bmatrix}}_{Z_{bus(new)}} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix}$$

2. Adding a new bus to an existing bus through an impedance.



A new bus p is added to an existing bus k through an impedance Z_b.
Since I_p is flowing into the network, the voltage at all the node will increase.
The voltage at the node ⓚ will be

$$
\begin{aligned}
V_k &= Z_{k1} I_1 + \cdots + Z_{kk}(I_k + I_p) + \cdots + Z_{kn} I_n \\
&= Z_{k1} I_1 + \cdots + Z_{kk} I_k + \cdots + Z_{kn} I_n + Z_{kk} I_p
\end{aligned}
$$

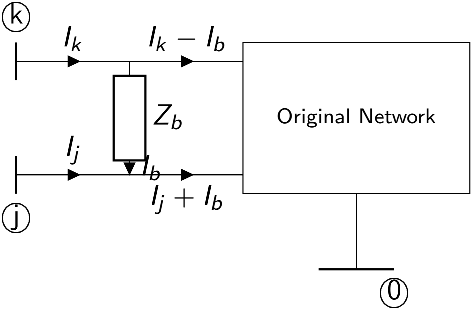
The voltage at the node ⓟ will be

$$V_p = V_k + I_p Z_b = Z_{k1} I_1 + \cdots + Z_{kk} I_k + \cdots + Z_{kn} I_n + (Z_{kk} + Z_b) I_p$$

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \\ V_p \end{bmatrix} = \underbrace{\begin{bmatrix} Z_{11} & \cdots & Z_{1n} & Z_{1k} \\ \vdots & \ddots & \vdots & \vdots \\ Z_{n1} & \cdots & Z_{nn} & Z_{nk} \\ Z_{k1} & \cdots & Z_{kn} & Z_{kk} + Z_b \end{bmatrix}}_{Z_{bus(new)}} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_p \end{bmatrix}$$

3. Adding an impedance between an existing bus and the reference bus
   - Add a new bus p temporarily to an existing bus k through an impedance Z_b.
   - The modified Z_{bus} will be $(n+1) \times (n+1)$.
   - Short-circuit the node p to the reference bus by letting V_p equal to zero.
   - Eliminate the node p using the Kron-reduction.

4. Adding an impedance between two existing buses



The voltage of the bus (1) will be

$$
\begin{aligned}
V_1 &= Z_{11}I_1 + \cdots + Z_{1j}(I_j + I_b) + Z_{1k}(I_k - I_b) + \cdots + Z_{1n}I_n \\
&= Z_{11}I_1 + \cdots + Z_{1j}I_j + Z_{1k}I_k + \cdots + Z_{1n}I_n + (Z_{1j} - Z_{1k})I_b
\end{aligned}
$$

Similarly, at buses (j) and (k)

$$
V_j = Z_{j1}I_1 + \cdots + Z_{jj}I_j + Z_{jk}I_k + \cdots + Z_{jn}I_n + (Z_{jj} - Z_{jk})I_b
$$

$$V_k = Z_{k1}I_1 + \cdots + Z_{kj}I_j + Z_{kk}I_k + \cdots + Z_{kn}I_n + (Z_{kj} - Z_{kk})I_b$$

Since I_b is unknown,

$$V_k - V_j = I_b Z_b$$

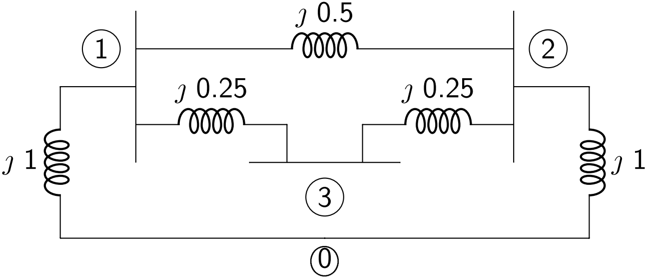
$$0 = (Z_{j1} - Z_{k1})I_1 + \cdots + (Z_{jj} - Z_{kj})I_j + (Z_{jk} - Z_{kk})I_k + \cdots + (Z_{jn} - Z_{kn})I_n + (Z_{jj} + Z_{kk} - 2Z_{jk} + Z_b)I_b$$

$$\begin{bmatrix} V_1 \\ \vdots \\ V_n \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & \cdots & Z_{1n} & (Z_{1j} - Z_{1k}) \\ \vdots & \ddots & \vdots & \vdots \\ Z_{n1} & \cdots & Z_{nn} & (Z_{nj} - Z_{nk}) \\ (Z_{j1} - Z_{k1}) & \cdots & (Z_{jn} - Z_{jk}) & Z_{bb} \end{bmatrix} \begin{bmatrix} I_1 \\ \vdots \\ I_n \\ I_b \end{bmatrix}$$

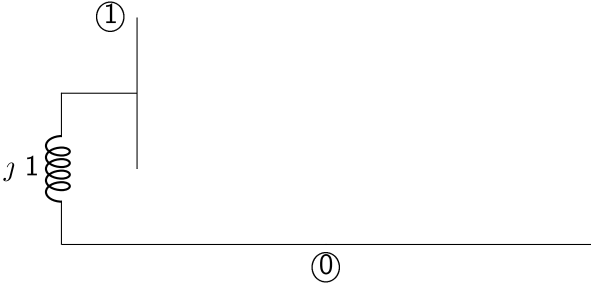
where $Z_{bb} = Z_{jj} + Z_{kk} - 2Z_{jk} + Z_b = Z_{th,\,jk} + Z_b$.

- Eliminate the last row and last column using the Kron-reduction.

Example 3:

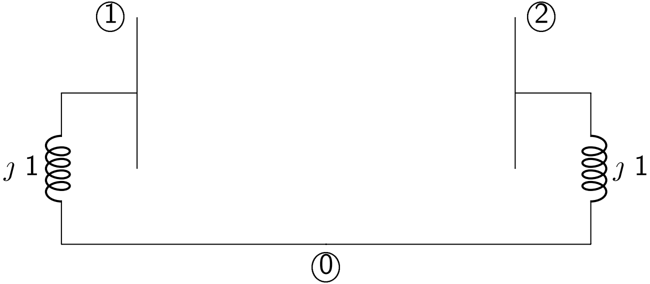


1. Let us add $\jmath 1$ from the reference bus to the bus ①.



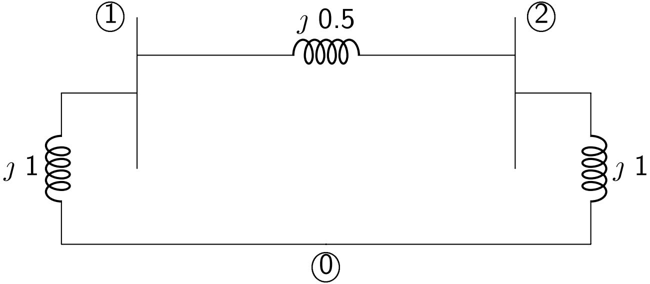
$$Z_{bus} = \begin{bmatrix} 0 & 0 \\ 0 & \jmath 1 \end{bmatrix} = [\jmath 1]$$

2. Let us add $j1$ from the reference bus to the bus ②.



$$Z_{bus} = \begin{bmatrix} j1 & 0 \\ 0 & j1 \end{bmatrix}$$

3. Let us add $\jmath 0.5$ between ① and ②.

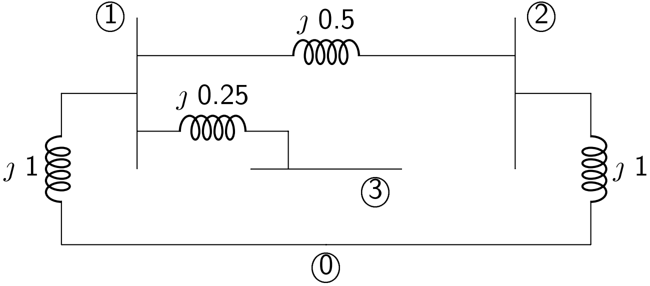


$$Z_{bus}^{temp} = \begin{bmatrix} \jmath 1 & 0 & \jmath 1 \\ 0 & \jmath 1 & -\jmath 1 \\ \jmath 1 & -\jmath 1 & \jmath 2.5 \end{bmatrix}$$

On eliminating the last row and last columns using the Kron-reduction,

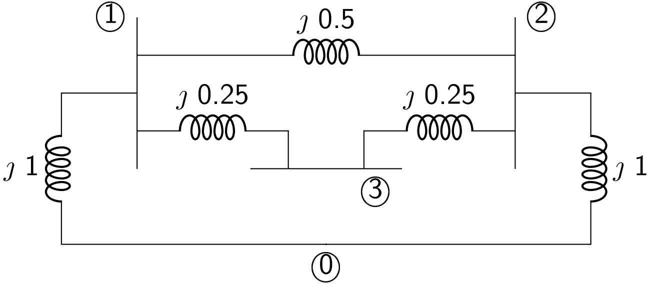
$$Z_{bus} = \begin{bmatrix} \jmath 1 - \dfrac{\jmath 1 \times \jmath 1}{\jmath 2.5} & 0 - \dfrac{\jmath 1 \times -\jmath 1}{\jmath 2.5} \\ 0 - \dfrac{\jmath 1 \times -\jmath 1}{\jmath 2.5} & \jmath 1 - \dfrac{-\jmath 1 \times -\jmath 1}{\jmath 2.5} \end{bmatrix} = \begin{bmatrix} \jmath 0.6 & \jmath 0.4 \\ \jmath 0.4 & \jmath 0.6 \end{bmatrix}$$

4. Let us add $\jmath 0.25$ between ① and a new bus ③.



$$Z_{bus} = \begin{bmatrix} \jmath 0.6 & \jmath 0.4 & \jmath 0.6 \\ \jmath 0.4 & \jmath 0.6 & \jmath 0.4 \\ \jmath 0.6 & \jmath 0.4 & \jmath 0.85 \end{bmatrix}$$

5. Let us add $j0.25$ between ② and ③.



$$Z_{bus}^{temp} = \begin{bmatrix} j0.6 & j0.4 & j0.6 & -j0.2 \\ j0.4 & j0.6 & j0.4 & j0.2 \\ j0.6 & j0.4 & j0.85 & -j0.45 \\ -j0.2 & j0.2 & -j0.45 & j0.9 \end{bmatrix}$$

On eliminating the last row and last column using the Kron-reduction,

$$Z_{bus} = \begin{bmatrix} \jmath 0.6 - \frac{-\jmath 0.2 \times -\jmath 0.2}{\jmath 0.9} & \jmath 0.4 - \frac{-\jmath 0.2 \times \jmath 0.2}{\jmath 0.9} & \jmath 0.6 - \frac{-\jmath 0.2 \times -\jmath 0.45}{\jmath 0.9} \\ \jmath 0.4 - \frac{-\jmath 0.2 \times \jmath 0.2}{\jmath 0.9} & \jmath 0.6 - \frac{\jmath 0.2 \times \jmath 0.2}{\jmath 0.9} & \jmath 0.4 - \frac{-\jmath 0.2 \times -\jmath 0.45}{\jmath 0.9} \\ \jmath 0.6 - \frac{-\jmath 0.2 \times -\jmath 0.45}{\jmath 0.9} & \jmath 0.4 - \frac{\jmath 0.2 \times -\jmath 0.45}{\jmath 0.9} & \jmath 0.85 - \frac{-\jmath 0.45 \times -\jmath 0.45}{\jmath 0.9} \end{bmatrix}$$

$$Z_{bus} = \begin{bmatrix} \jmath 0.5556 & \jmath 0.4444 & \jmath 0.5 \\ \jmath 0.4444 & \jmath 0.5556 & \jmath 0.5 \\ \jmath 0.5 & \jmath 0.5 & \jmath 0.6250 \end{bmatrix}$$

This can also be formed by inverting Y_{bus} matrix of the network.

$$Y_{bus} = \begin{bmatrix} -\jmath 7 & \jmath 2 & \jmath 4 \\ \jmath 2 & -\jmath 7 & \jmath 4 \\ \jmath 4 & \jmath 4 & -\jmath 8 \end{bmatrix}$$

$$Z_{bus} = Y_{bus}^{-1}$$

Calculation of Z_{bus} elements from Y_{bus} : If all the elements of Z_{bus} are not needed, the required elements can be calculated from Y_{bus}.

$$Y_{bus} Z_{bus} = I$$

$$Y_{bus} Z_{bus}^{(m)} = \begin{bmatrix} 0 \\ \vdots \\ 1_m \\ \vdots \\ 0 \end{bmatrix}$$

where $Z_{bus}^{(m)}$ is the m^{th} column of Z_{bus}. If the triangular factors are available,

$$LU Z_{bus}^{(m)} = \begin{bmatrix} 0 \\ \vdots \\ 1_m \\ \vdots \\ 0 \end{bmatrix}$$

The elements in the column vector $Z_{bus}^{(m)}$ can be found by forward elimination and back substitution.

Thevenin's Theorem and Z_{bus}:

Let us consider an n node network.

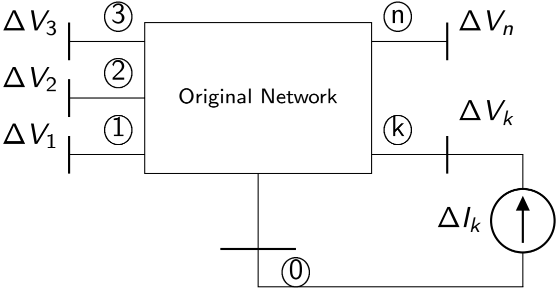
$$\mathrm{V}^0 = \mathrm{Z}_{bus}\mathrm{I}^0$$

where V^0 denotes the voltages of buses due to the bus currents I^0. When the bus currents are changed from their initial values to new values $(\mathrm{I}^0 + \Delta\mathrm{I})$,

$$\mathrm{V} = \mathrm{Z}_{bus}(\mathrm{I}^0 + \Delta\mathrm{I}) = \underbrace{\mathrm{Z}_{bus}\mathrm{I}^0}_{\mathrm{V}^0} + \underbrace{\mathrm{Z}_{bus}\Delta\mathrm{I}}_{\Delta\mathrm{V}}$$

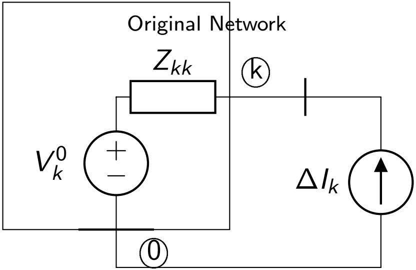
Let us assume that there is a change in the current injection only at bus ⓚ.

$$\begin{bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_k \\ \vdots \\ \Delta V_n \end{bmatrix} = \begin{bmatrix} Z_{11} & \cdots & Z_{1k} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ Z_{k1} & \cdots & Z_{kk} & \cdots & Z_{kn} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ Z_{n1} & \cdots & Z_{nk} & \cdots & Z_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \Delta I_k \\ \vdots \\ 0 \end{bmatrix}$$



$$\begin{bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_k \\ \vdots \\ \Delta V_n \end{bmatrix} = \begin{bmatrix} Z_{1k} \\ \vdots \\ Z_{kk} \\ \vdots \\ Z_{nk} \end{bmatrix} \Delta I_k$$

$$V_k = V_k^0 + Z_{kk} \Delta I_k$$



The Thevenin impedance at bus ⓚ is

$$Z_{th} = Z_{kk}$$

In a similar manner, we can determine the Thevenin impedance between any two buses ⓙ and ⓚ of the network.
Let us assume that there are current injections from ⓙ and ⓚ.

$$\begin{bmatrix} \Delta V_1 \\ \vdots \\ \Delta V_j \\ \Delta V_k \\ \vdots \\ \Delta V_n \end{bmatrix} = \begin{bmatrix} Z_{11} & \cdots & Z_{1j} & Z_{1k} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Z_{j1} & \cdots & Z_{jj} & Z_{jk} & \cdots & Z_{jn} \\ Z_{k1} & \cdots & Z_{kj} & Z_{kk} & \cdots & Z_{kn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Z_{n1} & \cdots & Z_{nj} & Z_{nk} & \cdots & Z_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \Delta I_j \\ \Delta I_k \\ \vdots \\ 0 \end{bmatrix}$$

The bus voltages at ⓙ and ⓚ are

$$V_j = V_j^0 + \Delta V_j = V_j^0 + Z_{jj} \Delta I_j + Z_{jk} \Delta I_k$$

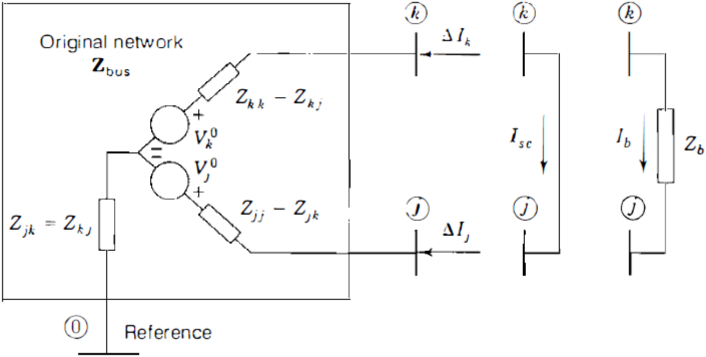
$$V_k = V_k^0 + \Delta V_k = V_k^0 + Z_{kj} \Delta I_j + Z_{kk} \Delta I_k$$

By adding and subtracting $Z_{jk}\Delta I_j$ in the first equation and $Z_{kj}\Delta I_k$ in the second equation,

$$V_j = V_j^0 + (Z_{jj} - Z_{jk})\Delta I_j + Z_{jk}(\Delta I_j + \Delta I_k)$$

$$V_k = V_k^0 + Z_{kj}(\Delta I_j + \Delta I_k) + (Z_{kk} - Z_{kj})\Delta I_k$$

The above equations can be represented as follows:



$$Z_{th,jk} = \frac{V_k^0 - V_j^0}{I_{sc}}$$

Since $I_{sc} = \Delta I_j = -\Delta I_k$ and $V_j - V_k = 0$,

$$0 = V_j^0 - V_k^0 + (Z_{jj} - Z_{jk})I_{sc} - (Z_{kk} - Z_{kj})(-I_{sc})$$

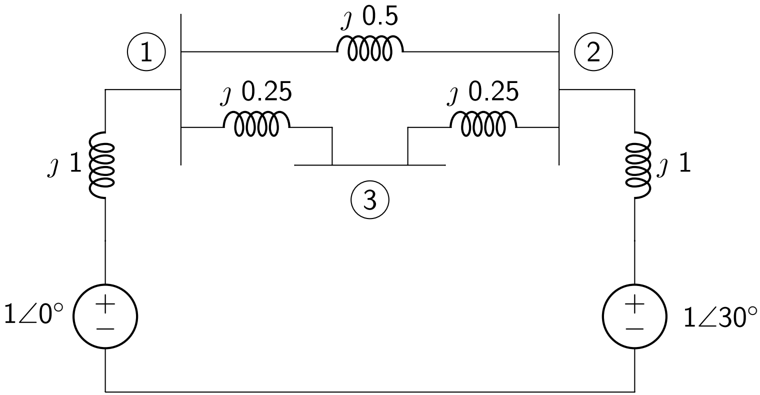
$$V_k^0 - V_j^0 = (Z_{jj} + Z_{kk} - 2Z_{kj})I_{sc}$$

$$Z_{th,jk} = Z_{jj} + Z_{kk} - 2Z_{kj}$$

The branch current I_b is given by

$$I_b = \frac{V_k - V_j}{Z_b} = \frac{V_k^0 - V_j^0}{Z_{th,jk} + Z_b}$$

Example 4: Consider the reactance network.



1. Find the voltages at each bus.
2. Find the voltage at each bus after connecting a capacitor having a reactance of 5 p.u. between ③ and ⓪.

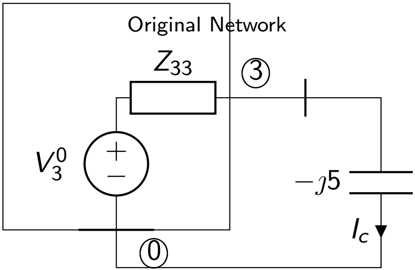
1.

$$\mathrm{V}_{bus}^{0} = \mathrm{Z}_{bus}\mathrm{I}_{bus}^{0}$$

$$\begin{bmatrix} V_1^0 \\ V_2^0 \\ V_3^0 \end{bmatrix} = \begin{bmatrix} j0.5556 & j0.4444 & j0.5 \\ j0.4444 & j0.5556 & j0.5 \\ j0.5 & j0.5 & j0.6250 \end{bmatrix} \begin{bmatrix} 1\angle{-90^\circ} \\ 1\angle{-60^\circ} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} V_1^0 \\ V_2^0 \\ V_3^0 \end{bmatrix} = \begin{bmatrix} 0.9663\angle{13.3^\circ} \\ 0.9664\angle{16.7^\circ} \\ 0.966\angle{15^\circ} \end{bmatrix}$$

2. To find change in voltage at each bus due to a capacitor:



$$I_c = \frac{V_3^0}{Z_{33} - \jmath X_c} = \frac{0.966\underline{/15^\circ}}{\jmath 0.6250 - \jmath 5} = 0.22\underline{/105^\circ}$$

$$\begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \Delta V_3 \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -I_c \end{bmatrix}$$

$$\begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \Delta V_3 \end{bmatrix} = \begin{bmatrix} \jmath 0.5 \\ \jmath 0.5 \\ \jmath 0.6250 \end{bmatrix} (-0.22\underline{/105^\circ}) = \begin{bmatrix} 0.11\underline{/15^\circ} \\ 0.11\underline{/15^\circ} \\ 0.1375\underline{/15^\circ} \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1^0 \\ V_2^0 \\ V_3^0 \end{bmatrix} + \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \\ \Delta V_3 \end{bmatrix} = \begin{bmatrix} 1.0765\underline{/13.5^\circ} \\ 1.0767\underline{/16.5^\circ} \\ 1.104\underline{/15^\circ} \end{bmatrix}$$

This example shows how adding a capacitor at a bus causes a rise in bus voltages.