Introduction

- Sinusoid means both cosine and sine waveforms.
- Sinusoidal voltage and current are referred to ac voltage and ac current respectively.
- The voltage available at our home, office and everywhere is sinusoid.

The single phase voltage

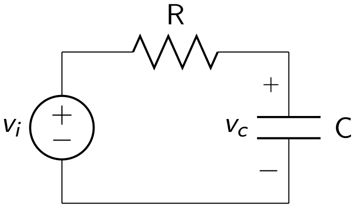
$$v_i = V_m \sin(\omega t)$$

where v_i - instantaneous voltage
V_m - maximum voltage. $\omega = 2\pi f$ - angular frequency (rad/sec).
However, the voltage available at our outlet is 230 V (rms).

$$V = \frac{V_m}{\sqrt{2}} \text{ (for sinusoid)}$$

It is root mean squared (rms) voltage. It is also a DC equivalent voltage.

RC Circuit



$$v_i = V_m \sin(\omega t)$$

By KVL,

$$RC\frac{dv_c}{dt} + v_c = V_m \sin(\omega t)$$

Let us solve this differential equation using the same steps.

- Homogeneous solution: To find this, set v_i to zero.

$$RC\frac{dv_{cH}}{dt} + v_{cH} = 0$$

As we have already solved in DC Transients,

$$\boxed{V_{cH} = Ae^{(-t/RC)}}$$

- Particular solution:

$$RC\frac{dv_{cP}}{dt} + v_{cP} = V_m \sin(\omega t)$$

Guess any solution that satisfies this equation.

1. First try. Let $v_{cP} = A$. It will not satisfy.
2. Second try. Let $v_{cP} = A\sin(\omega t)$. It will not also satisfy.
3. Third try. Let $v_{cP} = A\sin(\omega t + \phi)$. It will work.

On substituting this in the above equation,

$$RCA\omega\cos(\omega t + \phi) + A\sin(\omega t + \phi) = V_m \sin(\omega t)$$

$$RCA\omega(\cos(\omega t)\cos\phi - \sin\phi\sin(\omega t)) +$$

$$A\sin(\omega t)\cos\phi + A\sin\phi\cos(\omega t) = V_m \sin(\omega t)$$

$$\vdots$$

$$A = \frac{V_m}{\sqrt{1 + (\omega CR)^2}}$$

$$\phi = -\tan^{-1}(\omega CR)$$

$$\boxed{\therefore v_{cP} = \frac{V_m}{\sqrt{1 + (\omega CR)^2}} \sin(\omega t - \tan^{-1}(\omega CR))}$$

- Total solution:

$$v_c(t) = Ae^{(-t/RC)} + \frac{V_m}{\sqrt{1 + (\omega CR)^2}} \sin(\omega t - \tan^{-1}(\omega CR))$$

- To find constant: Use an initial condition.

First part is the transient response. Second part is the steady state response.

- Transient response is also called as natural response.
- It is always decaying exponential irrespective of DC or AC excitation.
- Forced (steady state) response is now sinusoid with a phase shift because excitation is sinusoid.
- If you give a sinusoid of one frequency to a linear circuit, the stead state response will be sinusoid of the same frequency with or without phase shift.

Since the transient response dies out, we are interested in **sinusoidal steady state response**.

$$v_c(t) = V_C \sin(\omega t + \phi)$$

where $V_C = \dfrac{V_m}{\sqrt{1 + (\omega CR)^2}}$ and $\phi = -\tan^{-1}(\omega CR)$.

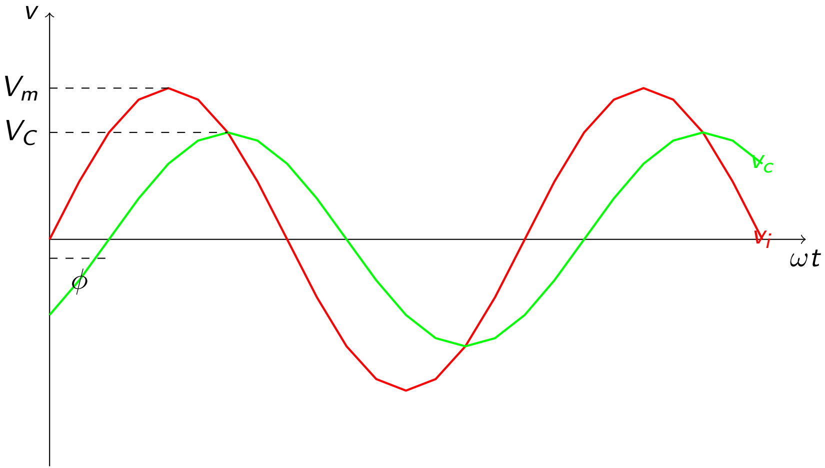


Figure: Sinusoidal steady state response - RC Circuit

Simple Approach

Let us a simple approach to find the sinusoidal steady state response[1] of a linear circuit.

$$V_m \sin(\omega t) \rightsquigarrow A \sin(\omega t + \phi)$$

$$V_m \cos(\omega t) \rightsquigarrow A \cos(\omega t + \phi)$$

Since the circuit is linear,

$$\jmath V_m \sin(\omega t) \rightsquigarrow \jmath A \sin(\omega t + \phi)$$

$$V_m \cos(\omega t) + \jmath V_m \sin(\omega t) \rightsquigarrow A \cos(\omega t + \phi) + \jmath A \sin(\omega t + \phi)$$

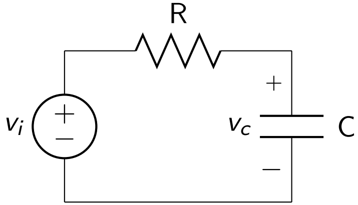
By Euler's identity,

$$V_m e^{\jmath \omega t} \rightsquigarrow A e^{\jmath(\omega t + \phi)}$$

If your input is sine, then take imaginary component of the response.

[1] Steady state response follows superposition

Let us consider the same circuit.



Let us apply $v_i = V_m e^{\jmath\omega t}$ (mathematically possible).
By KVL,

$$RC\frac{dv_c}{dt} + v_c = V_m e^{\jmath\omega t}$$

If we apply this voltage, the steady state response v_c would be $Ae^{\jmath(\omega t+\phi)}$.

$$RCA\jmath\omega e^{\jmath(\omega t+\phi)} + Ae^{\jmath(\omega t+\phi)} = V_m e^{\jmath\omega t}$$

$$(RCA\jmath\omega + 1)Ae^{\jmath\phi} = V_m$$

$$Ae^{\jmath\phi} = \frac{V_m}{(RCA\jmath\omega + 1)}$$

$$Ae^{\jmath\phi} = \frac{V_m}{\sqrt{1 + (\omega CR)^2}} e^{\jmath(-\tan^{-1}(\omega CR))}$$

It can be written in polar form as follows.

$$A\angle\phi = \frac{V_m}{\sqrt{1 + (\omega CR)^2}} \angle - \tan^{-1}(\omega CR)$$

Since the excitation is sine, we have to take imaginary part of the above complex response.

$$v_c(t) = \frac{V_m}{\sqrt{1 + (\omega CR)^2}} \sin(\omega t - \tan^{-1}(\omega CR))$$

If the excitation were cosine, we would take the real part of the complex response.

Phasor Approach

- This is simpler than previous approach.
- We found $e^{\jmath\omega t}$ as redundant since the frequency of sinusoidal steady state response is same as the supply frequency in a linear circuit.

$$V_m e^{\jmath\omega t} \rightsquigarrow V_C e^{\jmath\omega t+\phi}$$

$$V_m e^{\jmath\omega t} e^{\jmath 0} \rightsquigarrow V_C e^{\jmath\omega t} e^{\jmath\phi}$$

Since $e^{\jmath\omega t}$ appears in both terms, it can be removed.
Let us apply.

$$V_m e^{\jmath 0} \rightsquigarrow V_C e^{\jmath\phi}$$

$$V_m \angle 0 \rightsquigarrow V_C \angle \phi$$

$$\mathbf{V_m} \rightsquigarrow \mathbf{V_C}$$

where $\mathbf{V_m}$ and $\mathbf{V_C}$ are complex numbers and called as **phasors**.
Phasor is a complex number that contains magnitude and phase angle information.

Resistor

In time domain,

$$v(t) = i(t)R$$

Let us apply complex current through resistor.

$$i(t) = I_m e^{\jmath \omega t}$$

The voltage response will be

$$v(t) = V_m e^{\jmath(\omega t + \phi)}$$

$$V_m e^{\jmath(\omega t + \phi)} = I_m e^{\jmath \omega t} R$$

$$V_m e^{\jmath \phi} = I_m e^{\jmath 0} R$$

$$V_m \angle \phi = I_m \angle 0 \, R$$

In frequency domain,

$$\boxed{\mathbf{V_m} = \mathbf{I_m} R}$$

Since $\phi = 0$, voltage and current are in **phase**.

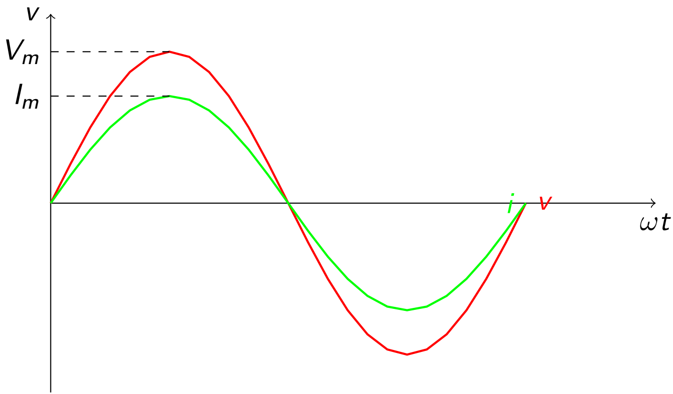


Figure: Voltage and current waveforms of Resistor

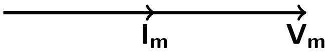


Figure: Phasor diagram of Resistor

Inductor

In time domain,

$$v(t) = L\frac{di}{dt}$$

Let us apply complex current through inductor.

$$i(t) = I_m e^{\jmath \omega t}$$

The voltage response will be

$$v(t) = V_m e^{\jmath(\omega t + \phi)}$$

$$V_m e^{\jmath(\omega t + \phi)} = \jmath \omega L I_m e^{\jmath \omega t}$$

$$V_m e^{\jmath \phi} = \jmath \omega L I_m e^{\jmath 0}$$

$$V_m \angle \phi = I_m \angle 0 \, \jmath \omega L$$

In frequency domain,

$$\boxed{\mathbf{V_m} = \jmath \omega L \mathbf{I_m}}$$

Since $\phi = 90$, voltage leads current by 90° in inductor.

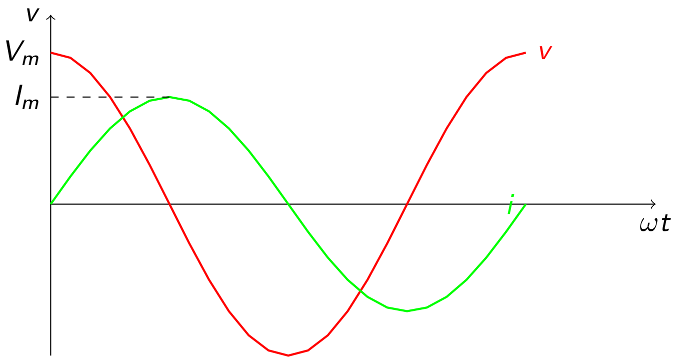


Figure: Voltage and current waveforms of Inductor

Figure: Phasor diagram of Inductor

Capacitor

In time domain,

$$i(t) = C\frac{dv}{dt}$$

Let us apply complex voltage across capacitor.

$$v(t) = V_m e^{\jmath \omega t}$$

The current will be

$$i(t) = I_m e^{\jmath(\omega t + \phi)}$$

$$I_m e^{\jmath(\omega t + \phi)} = \jmath \omega C V_m e^{\jmath \omega t}$$

$$I_m e^{\jmath \phi} = \jmath \omega C V_m e^{\jmath 0}$$

$$I_m \angle \phi = V_m \angle 0 \, \jmath \omega C$$

In frequency domain,

$$\boxed{\mathbf{V_m} = \frac{\mathbf{I_m}}{\jmath \omega C}}$$

Since $\phi = 90$, current leads voltage by 90° in inductor.

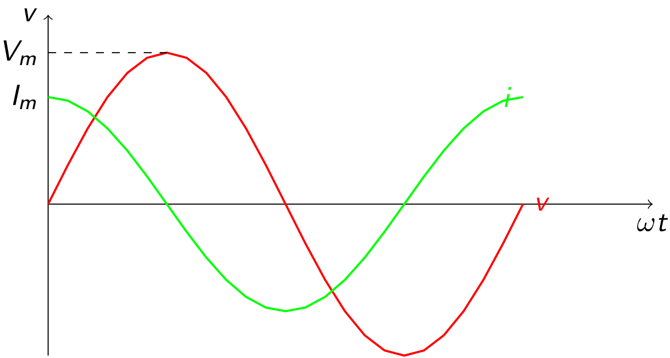


Figure: Voltage and current waveforms of Capacitor

Figure: Phasor diagram of Capacitor

1. In Resistor, there is no significant advantage in frequency domain.
2. In Inductor and Capacitor, the voltage and current are related by an algebraic equation in frequency domain. (This is great).
3. The ratio of complex voltage to complex current is **impedance (Z)**.It's unit is Ω.

$$\mathbf{Z}_R = R, \quad \mathbf{Z}_L = \jmath\omega L, \quad \mathbf{Z}_C = \frac{1}{\jmath\omega C}$$

   In general,

$$\mathbf{Z} = R + \jmath X$$

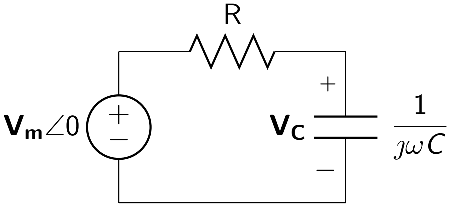
4. **Admittance** is the reciprocal of impedance and denoted by **Y**. It's unit is $\mho$.

$$\mathbf{Y}_R = \frac{1}{R}, \quad \mathbf{Y}_L = \frac{1}{\jmath\omega L}, \quad \mathbf{Y}_C = \jmath\omega C$$

   In general,

$$\mathbf{Y} = G + \jmath B$$

RC Circuit - Frequency domain



By voltage division,

$$\mathbf{V_C} = \mathbf{V_m}\angle 0 \frac{\frac{1}{\jmath\omega C}}{R + \frac{1}{\jmath\omega C}} = \frac{\mathbf{V_m}}{\sqrt{1+(\omega CR)^2}}\angle - \tan^{-1}(\omega CR)$$

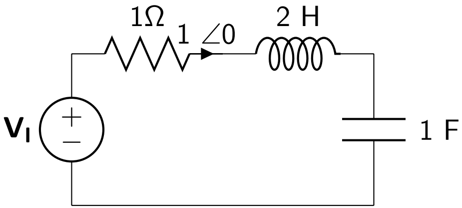
That's it. How simple it is.

$$|\mathbf{V_C}| = \frac{\mathbf{V_m}}{\sqrt{1+(\omega CR)^2}}, \quad \angle\mathbf{V_C} = -\tan^{-1}(\omega CR)$$

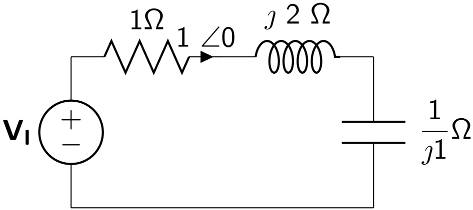
If you want, the response can be written in time domain.

RLC Circuit - Example

Find $\mathbf{V_I}$ and draw the phasor diagram. Assume $\omega = 1$ rad/sec.



In Frequency domain,



By KVL, (KVL and KCL are equally applicable in frequency domain too.)

$$\mathbf{V_I} = \mathbf{I}R + \mathbf{I}\jmath\omega L + \mathbf{I}\frac{1}{\jmath\omega C}$$

$$\mathbf{V_I} = (1 + \jmath 2 + \frac{1}{\jmath}) \times 1\angle 0$$

$$\mathbf{V_I} = \sqrt{2}\angle 45^\circ \text{ V}$$

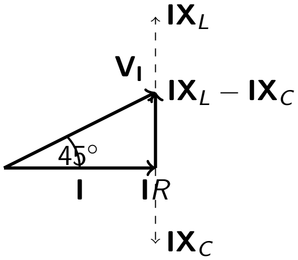


Figure: Phasor diagram taking **I** as reference