CS5101: Design and Analysis of Algorithms

Network Flows



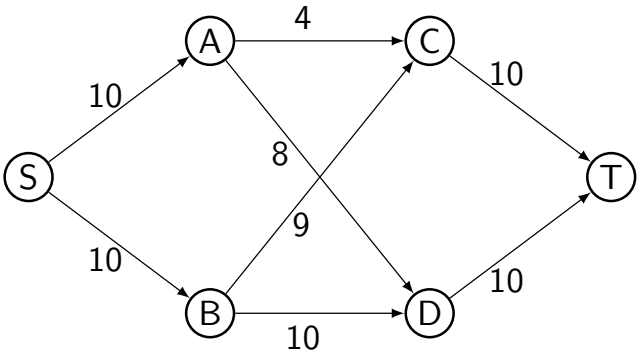
Arijit Mondal

Dept of CSE

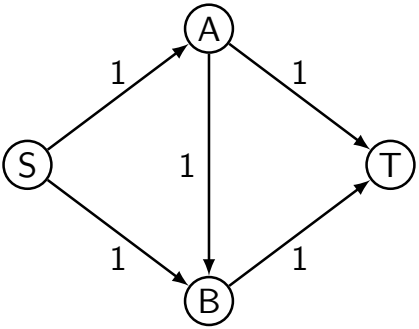
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Example



Example



Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction

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- Goal is to find maximum f for the given network G

Ford Fulkerson Method

- Steps: Ford-Fulkerson(G, s, t)
  1. Initialize flow f to 0
  2. while there exists an augmenting path p in the residual network G_f
  3. Augment flow f in path p
  4. return f

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  1. Initialize flow f to 0
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- For a flow network $G = (V, E)$ with source s, target t and a flow of f, consider a pair of vertices u, v, residual capacity will be

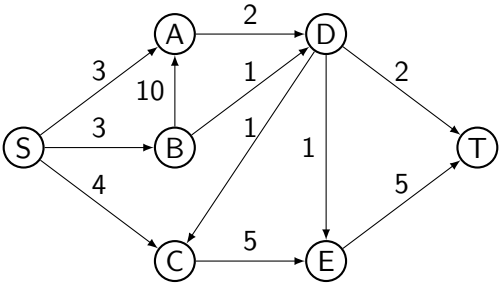
$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \end{cases}$$

Ford Fulkerson Method: pseudocode

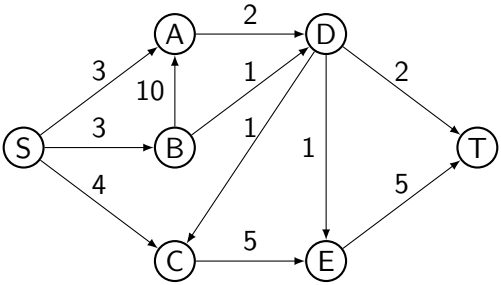
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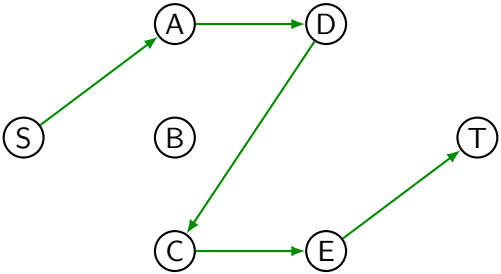
1. **for each** edge $(u, v) \in E$ **do** $(u, v).f = 0$
2. **while** there exists an augmenting path p in the residual network G_f
3. $\quad c_f(p) = \min\{c_f(u, v) : (u, v) \in p\}$
4. $\quad$ **for each** edge $(u, v) \in p$ **do**
5. $\quad\quad$ **if** $(u, v) \in E$ **then**
6. $\quad\quad\quad (u, v).f = (u, v).f + c_f(p)$
7. $\quad\quad$ **else**
8. $\quad\quad\quad (v, u).f = (v, u).f - c_f(p)$

Netwrok flow - 1

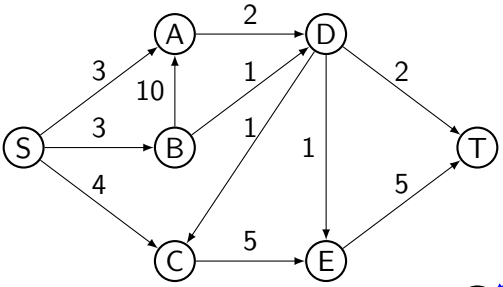


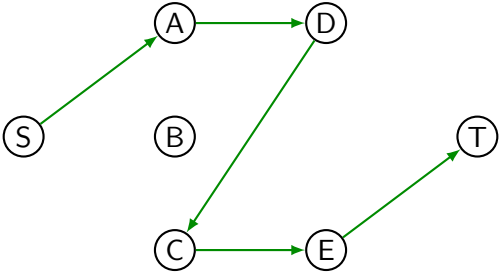
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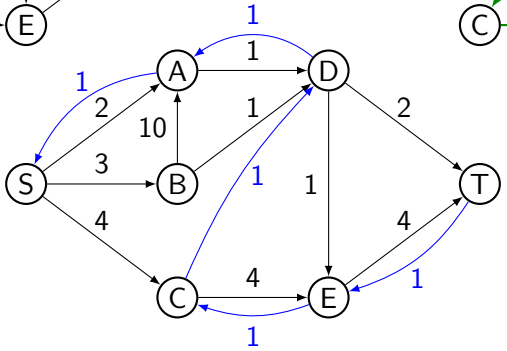




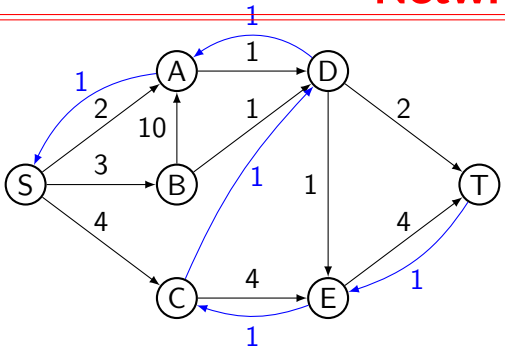
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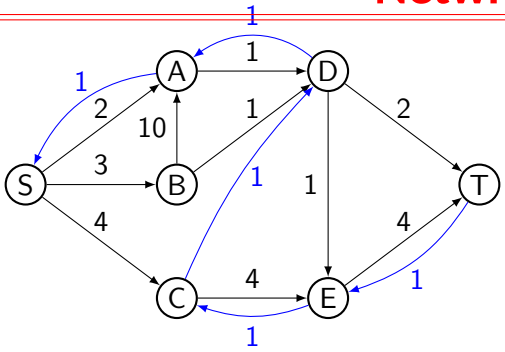


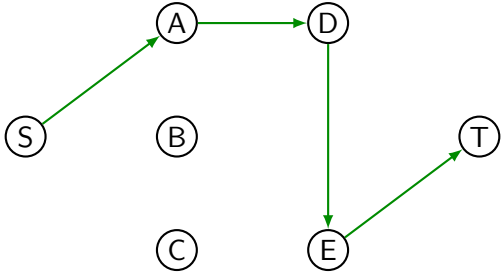


Netwrok flow - 2

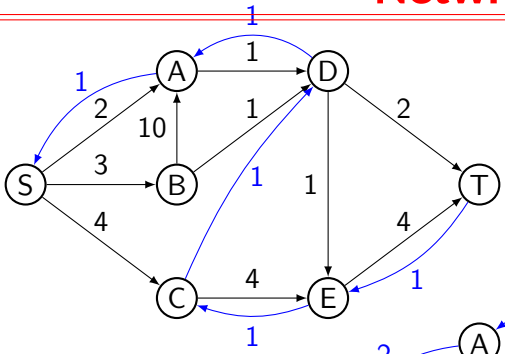


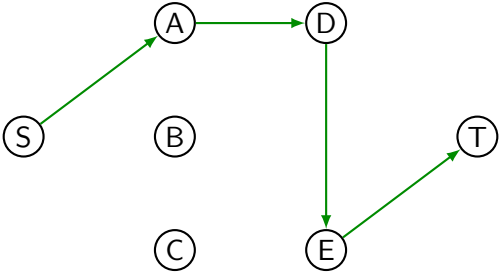
Netwrok flow - 2

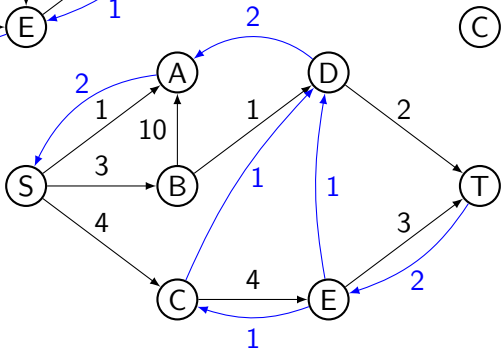




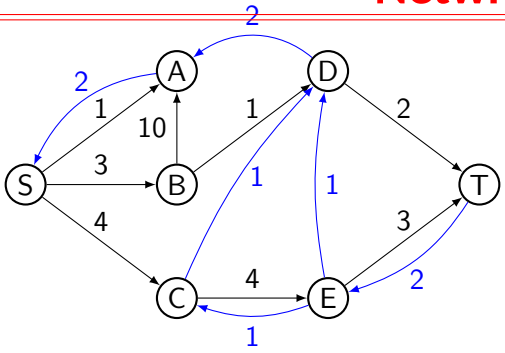
Netwrok flow - 2



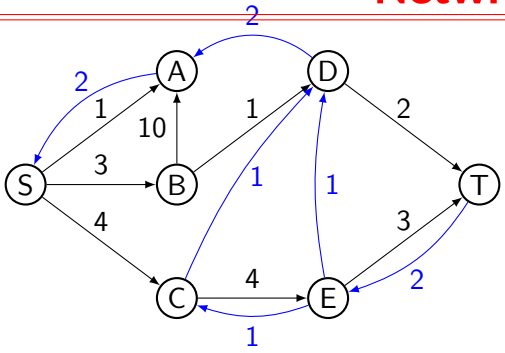


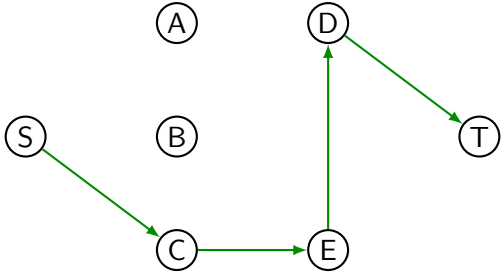


Netwrok flow - 3

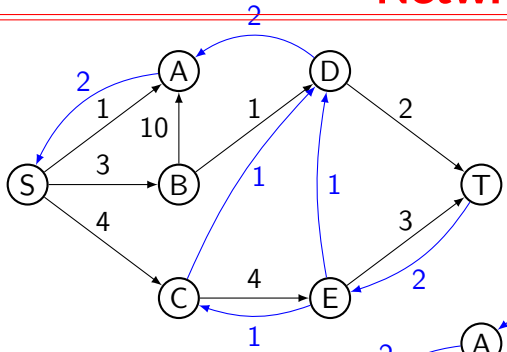


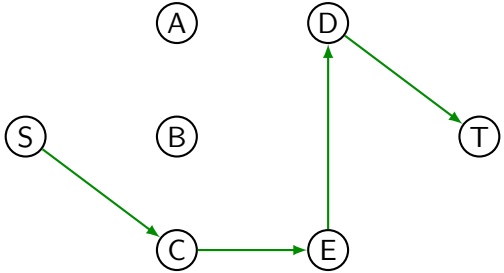
Netwrok flow - 3

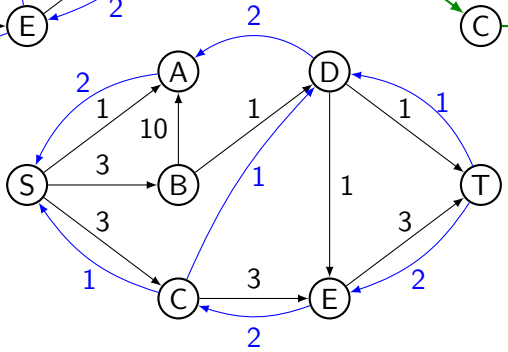




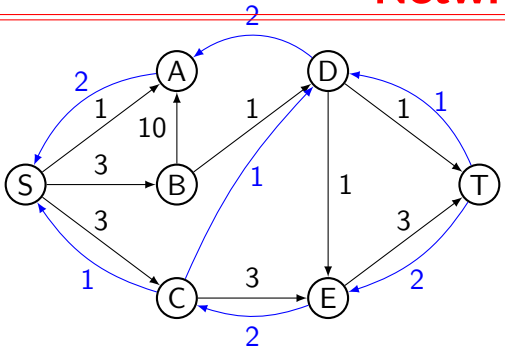
Netwrok flow - 3



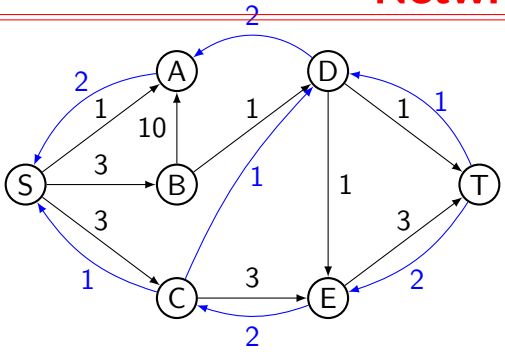


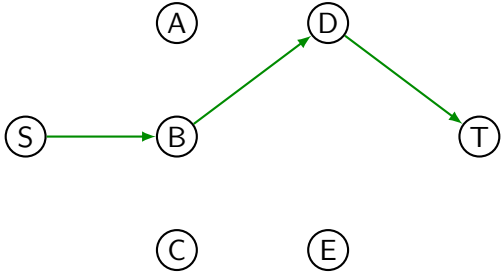


Netwrok flow - 4

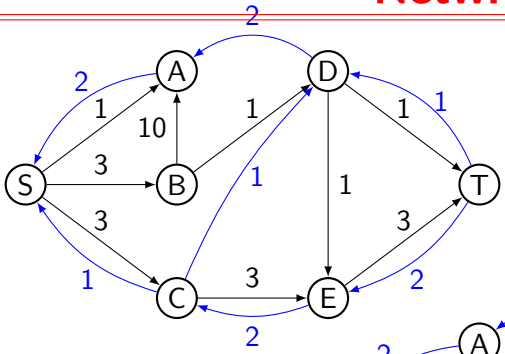


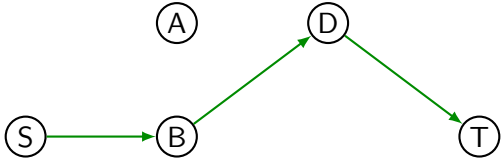
Netwrok flow - 4

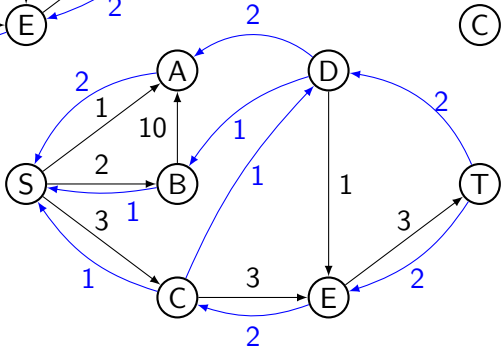




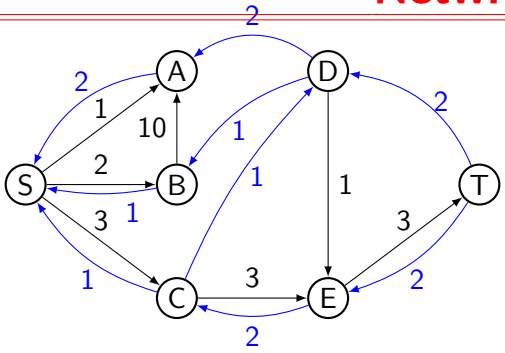
Netwrok flow - 4



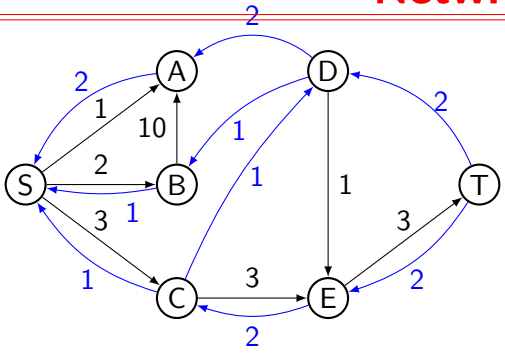


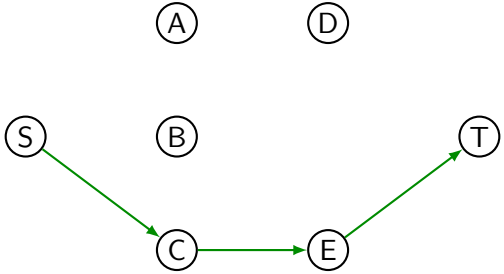


Netwrok flow - 5

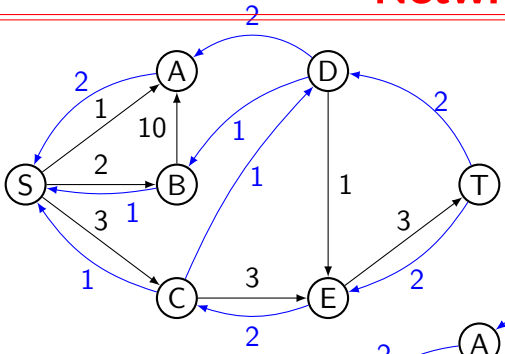


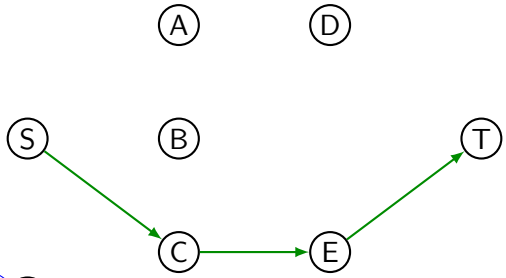
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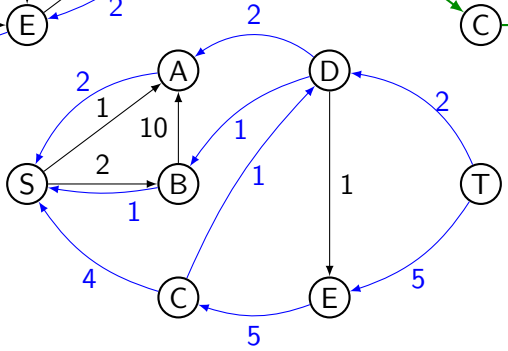




Netwrok flow - 5







Augmentation

- If f is a flow in G and f' is a flow in the corresponding residual network G_f, we define $f \uparrow f'$ the augmentation of flow f by f', to be a function from $V \times V$ to $\mathbb{R}$ defined by

$$(f \uparrow f') = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

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- Let $G = (V, E)$ be a flow network with source s and sink t and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then, $f \uparrow f' = |f| + |f'|$ holds
- To prove the above, we need to show the following:
  - $f \uparrow f'(u,v) \geq 0$ – Non-negative flow
  - $f \uparrow f'(u,v) \leq c(u,v)$ – Not exceeding the capacity
  - $f \uparrow f' = |f| + |f'|$

Proof - 1

- Non-negative flow

$$(f \uparrow f')(u, v) = f(u, v) + f'(u, v) - f'(v, u)$$

Proof - 1

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$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) \\ &\geq f(u,v) + f'(u,v) - f(u,v)\end{aligned}$$

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\begin{aligned}
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&= f'(u,v) \\
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\end{aligned}
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- Capacity constraint

$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) \\ &\leq f(u,v) + f'(u,v) \\ &\leq f(u,v) + c_f(u,v) \\ &= f(u,v) + c(u,v) - f(u,v) \\ &= c(u,v)\end{aligned}$$

Proof - 2

- We have,

$$\sum_{v \in V} (f \uparrow f')(u, v) - \sum_{v \in V} (f \uparrow f')(v, u)$$

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$$= \sum_{v \in V_l(u)} (f \uparrow f')(u, v) - \sum_{v \in V_e(u)} (f \uparrow f')(v, u) \qquad [V_l, V_e - \text{leaving, entering}]$$

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$$= \sum_{v \in V_l(u)}(f(u,v) + f'(u,v) - f'(v,u)) - \sum_{v \in V_e(u)}(f(v,u) + f'(v,u) - f'(u,v))$$

Proof - 2

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$$= \sum_{v \in V_l(u)}(f(u,v) + f'(u,v) - f'(v,u)) - \sum_{v \in V_e(u)}(f(v,u) + f'(v,u) - f'(u,v))$$

$$= \sum_{v \in V_l(u)} f(u,v) + \sum_{v \in V_l(u)} f'(u,v) - \sum_{v \in V_l(u)} f'(v,u) - \sum_{v \in V_e(u)} f(v,u) - \sum_{v \in V_e(u)} f'(v,u) + \sum_{v \in V_e(u)} f'(u,v)$$

Proof - 2

- We have,

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$$= \sum_{v \in V_l(u)} f(u,v) - \sum_{v \in V_e(u)} f(v,u) + \sum_{v \in V_l(u)} f'(u,v) + \sum_{v \in V_e(u)} f'(u,v) - \sum_{v \in V_l(u)} f'(v,u) - \sum_{v \in V_e(u)} f'(v,u)$$

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$$= \sum_{v \in V_l(u)} f(u,v) - \sum_{v \in V_e(u)} f(v,u) + \sum_{v \in V_l(u) \cup V_e(u)} f'(u,v) - \sum_{v \in V_l(u) \cup V_e(u)} f'(v,u)$$

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- Choose $u = s$

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- The capacity of the cut (S, T) is $c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$
- A minimum cut of a netwrok is a cut whose capacity is minimum over all cuts of the network

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$$\begin{aligned} |f| &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right) \\ &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \sum_{v \in V} f(u, v) - \sum_{u \in S - \{s\}} \sum_{v \in V} f(v, u)) \end{aligned}$$

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|f| &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right) \\
&= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \sum_{v \in V} f(u, v) - \sum_{u \in S - \{s\}} \sum_{v \in V} f(v, u)) \\
&= \sum_{v \in V} \left(f(s, v) + \sum_{u \in S - \{s\}} f(u, v) \right) - \sum_{v \in V} \left(f(v, s) + \sum_{u \in S - \{s\}} f(v, u) \right)
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&= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \sum_{v \in V} f(u, v) - \sum_{u \in S - \{s\}} \sum_{v \in V} f(v, u)) \\
&= \sum_{v \in V} \left(f(s, v) + \sum_{u \in S - \{s\}} f(u, v) \right) - \sum_{v \in V} \left(f(v, s) + \sum_{u \in S - \{s\}} f(v, u) \right) \\
&= \sum_{v \in V} \sum_{u \in S} f(u, v) - \sum_{v \in V} \sum_{u \in S} f(v, u)
\end{aligned}$$

Cut & Flow-3

- We have

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|f| &= \sum_{v \in V} \sum_{u \in S} f(u,v) - \sum_{v \in V} \sum_{u \in S} f(v,u) \qquad [V = S \cup T] \\
&= \sum_{v \in S} \sum_{u \in S} f(u,v) + \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) - \sum_{v \in T} \sum_{u \in S} f(v,u) \\
&= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) + \left(\sum_{v \in S} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) \right)
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&= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u) + \left(\sum_{v \in S} \sum_{u \in S} f(u,v) - \sum_{v \in S} \sum_{u \in S} f(v,u) \right) \\
&= \sum_{v \in T} \sum_{u \in S} f(u,v) - \sum_{v \in T} \sum_{u \in S} f(v,u)
\end{aligned}
$$

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- We have

$$
\begin{aligned}
|f| &= \sum_{v\in V}\sum_{u\in S} f(u,v) - \sum_{v\in V}\sum_{u\in S} f(v,u) \qquad [V = S \cup T] \\
&= \sum_{v\in S}\sum_{u\in S} f(u,v) + \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) + \left(\sum_{v\in S}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u)\right) \\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\
&= f(S,T)
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Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G

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$$\begin{aligned} |f| &= f(S,T) \\ &= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \end{aligned}$$

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$$\begin{aligned} |f| &= f(S,T) \\ &= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \\ &\leq \sum_{u\in S}\sum_{v\in T} f(u,v) \end{aligned}$$

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&\le \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v)
\end{aligned}
$$

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&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)
\end{aligned}
$$

Max-flow min-cut

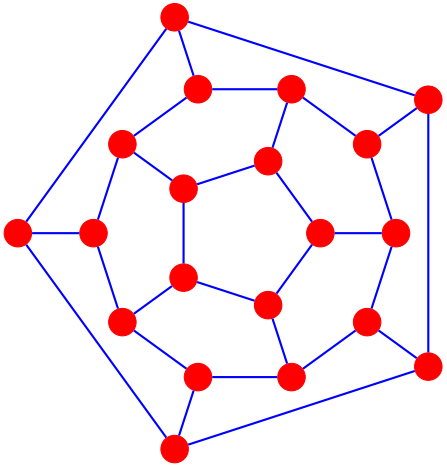
- If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:
  - f is a maximum flow in G
  - The residual network G_f contains no augmenting paths
  - $|f| = c(S, T)$ for some cut (S, T) of G

Edmonds-Karp algorithm

- Augmenting path with fewest edges needs to be chosen
- Breadth-first search can be used to find augmenting path in the residual network
- Time complexity becomes $O(VE^2)$

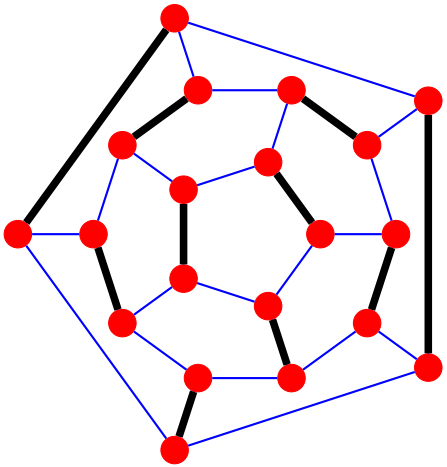
Matching

- Given an undirected graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a matching if each node of the graph appears *at most* one edge of M.



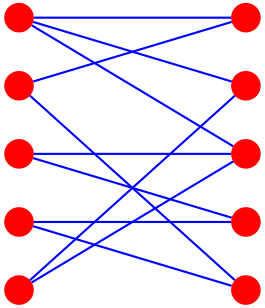
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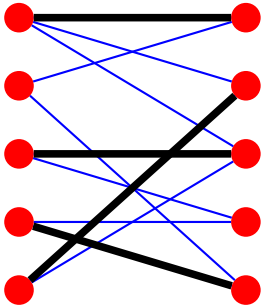
Bipartite matching

- A graph is bipartite if the nodes can be partitioned into two subsets X and Y such that every edge connects a node in X to a node in Y
- Given a bipartite graph $G = (X \cup Y, E)$, find a matching (M) that has the maximum cardinality ie., $|M|$ is maximum.



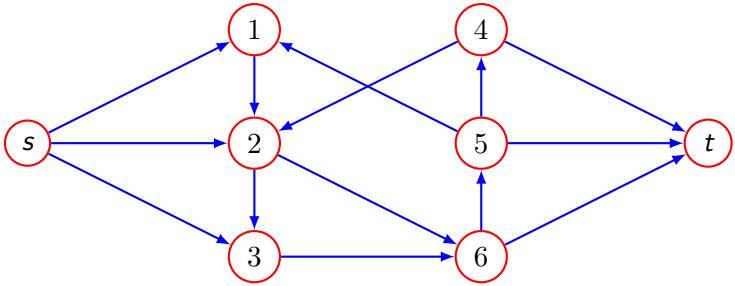
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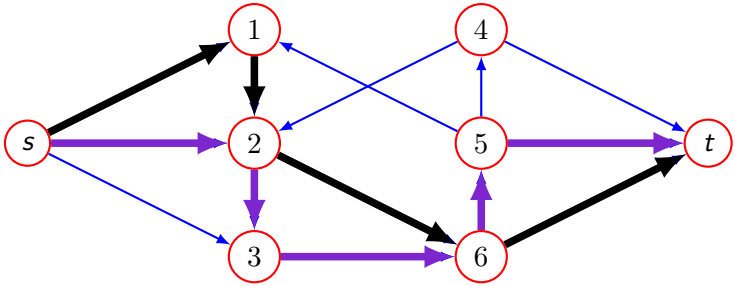
Edge-disjoint paths

- Two paths are edge-disjoint if they have no common edge. Given a directed graph $G = (V, E)$ and two nodes s and t, find the maximum number of edge-disjoint $s \rightsquigarrow t$ paths.



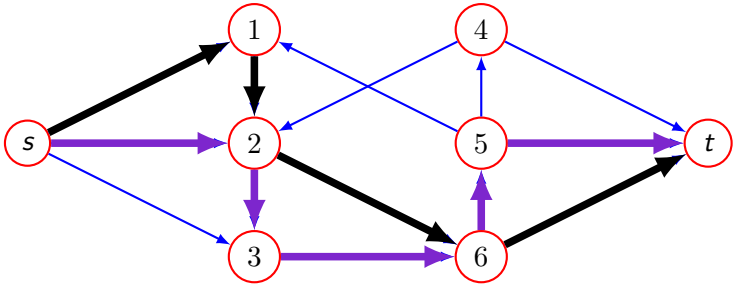
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Network connectivity

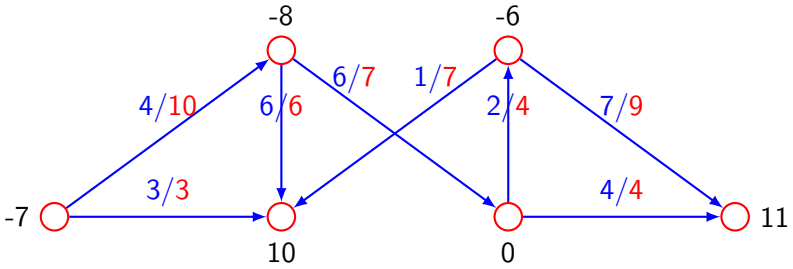
- Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.



Circulation with demands

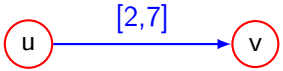
- Given a directed graph $V = (G, E)$ with non-negative edge capacities $c(e)$ and node supply and demands $d(v)$, a circulation is a function that satisfies
  - For each $e \in E$: $0 \leq f(e) \leq c(e)$ ($f(.)$ – flow along edge e)
  - For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$

  Does a circulation exist?
- $d(v) > 0$ - demand, $d(v) < 0$ - supply, $d(v) = 0$ - transshipment node



Circulation with lower bounds

- The problem is the same as previous one except that each edge has some lower bound on the flow. Hence, capacity along an edge will be specified as $[c_{lb}(u, v), c_{ub}(u, v)]$. What modifications are to be made in the graph to apply previous strategy?



Survey design

- Design a survey asking n_1 consumers about n_2 products that meets the following requirements, if possible.
  - Consumer i can survey about product j if they own it
  - Consumer i can be asked between c_i and c_i' questions
  - Ask between p_j and p_j' consumers about product j

Thank you!