CS514: Design and Analysis of Algorithms

Network Flow

Arijit Mondal

Dept of CSE

arijit@iitp.ac.in

https://www.iitp.ac.in/~arijit/

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction
- Each flow network contains a source (s) and a target (t) nodes
- In case of multiple source and target nodes, dummy source and target nodes can be added

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction
- Each flow network contains a source (s) and a target (t) nodes
- In case of multiple source and target nodes, dummy source and target nodes can be added
- A flow f is a real valued function $f : V \times V \rightarrow \mathbb{R}$

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction
- Each flow network contains a source (s) and a target (t) nodes
- In case of multiple source and target nodes, dummy source and target nodes can be added
- A flow f is a real valued function $f : V \times V \rightarrow \mathbb{R}$
- Capacity constraint: for all $u, v \in V$ we have $0 \leq f(u, v) \leq c(u, v)$, c is the capacity

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction
- Each flow network contains a source (s) and a target (t) nodes
- In case of multiple source and target nodes, dummy source and target nodes can be added
- A flow f is a real valued function $f: V \times V \rightarrow \mathbb{R}$
- Capacity constraint: for all $u, v \in V$ we have $0 \leq f(u, v) \leq c(u, v)$, c is the capacity
- Flow conservation: for all $u \in V - \{s, t\}$ we have $\sum_{v \in V} f(u, v) = \sum_{w \in V} f(w, u)$

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction
- Each flow network contains a source (s) and a target (t) nodes
- In case of multiple source and target nodes, dummy source and target nodes can be added
- A flow f is a real valued function $f : V \times V \rightarrow \mathbb{R}$
- Capacity constraint: for all $u, v \in V$ we have $0 \leq f(u, v) \leq c(u, v)$, c is the capacity
- Flow conservation: for all $u \in V - \{s, t\}$ we have $\sum_{v \in V} f(u, v) = \sum_{w \in V} f(w, u)$
- Flow in a network $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$

Network Flow

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. We further assume that if E contains an edge (u, v), then there is no edge (v, u) in the reverse direction
- Each flow network contains a source (s) and a target (t) nodes
- In case of multiple source and target nodes, dummy source and target nodes can be added
- A flow f is a real valued function $f : V \times V \to \mathbb{R}$
- Capacity constraint: for all $u, v \in V$ we have $0 \leq f(u, v) \leq c(u, v)$, c is the capacity
- Flow conservation: for all $u \in V - \{s, t\}$ we have $\sum_{v \in V} f(u, v) = \sum_{w \in V} f(w, u)$
- Flow in a network $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$
- Goal is to find maximum f for the given network G

Ford Fulkerson Method

- Steps: Ford-Fulkerson(G, s, t)

1. Initialize flow f to 0
2. while there exists an augmenting path p in the residual network G_f
3. Augment flow f in path p
4. return f

Ford Fulkerson Method

- Steps: Ford-Fulkerson(G, s, t)
  1. Initialize flow f to 0
  2. while there exists an augmenting path p in the residual network G_f
  3. Augment flow f in path p
  4. return f

- For a flow network $G = (V, E)$ with source s, target t and a flow of f, consider a pair of vertices u, v, residual capacity will be

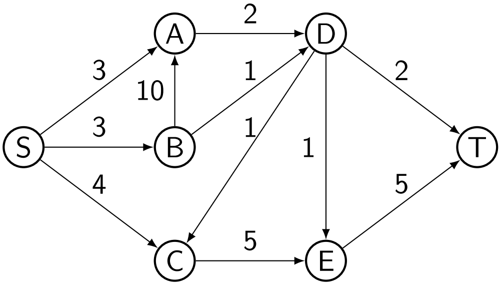
$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \end{cases}$$

Ford Fulkerson Method: pseudocode

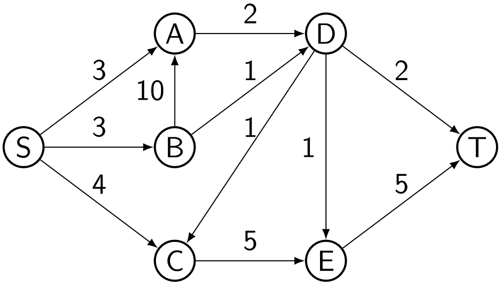
- Steps: Ford-Fulkerson(G, s, t)

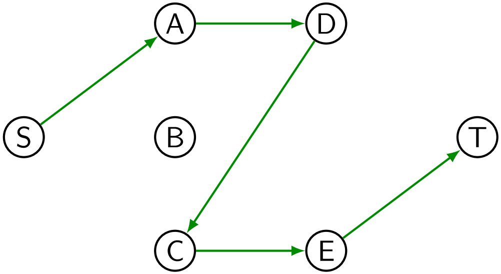
1. **for each** edge $(u, v) \in E$ **do** $(u, v).f = 0$
2. **while** there exists an augmenting path p in the residual network G_f
3. $\quad c_f(p) = \min\{c_f(u, v) : (u, v) \in p\}$
4. $\quad$ **for each** edge $(u, v) \in p$ **do**
5. $\quad\quad$ **if** $(u, v) \in E$ **then**
6. $\quad\quad\quad (u, v).f = (u, v).f + c_f(p)$
7. $\quad\quad$ **else**
8. $\quad\quad\quad (v, u).f = (v, u).f - c_f(p)$

Netwrok flow - 1

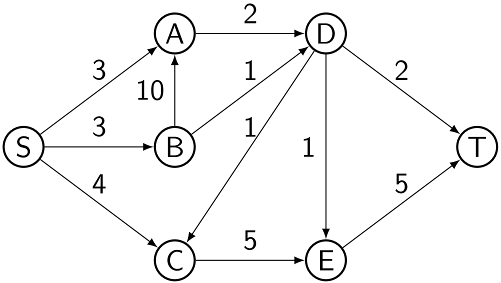


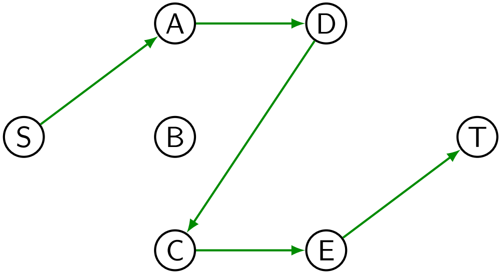
Netwrok flow - 1

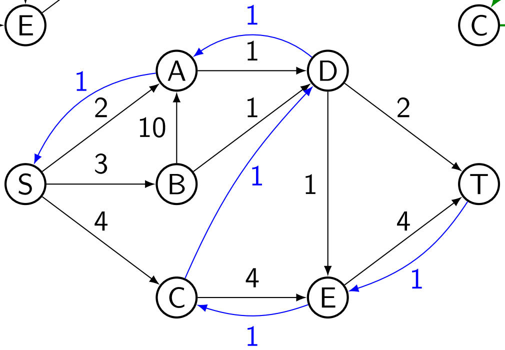




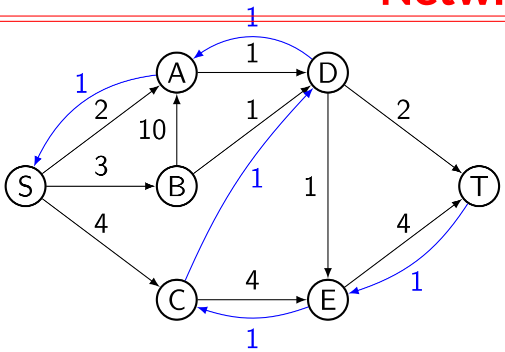
Netwrok flow - 1



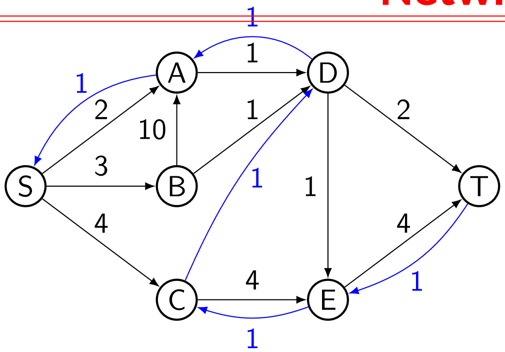


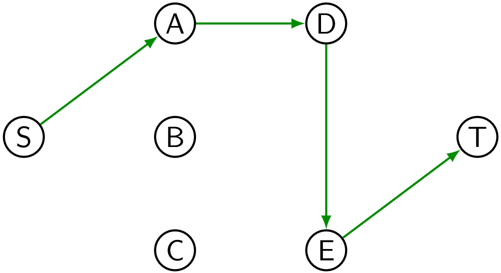


Netwrok flow - 2

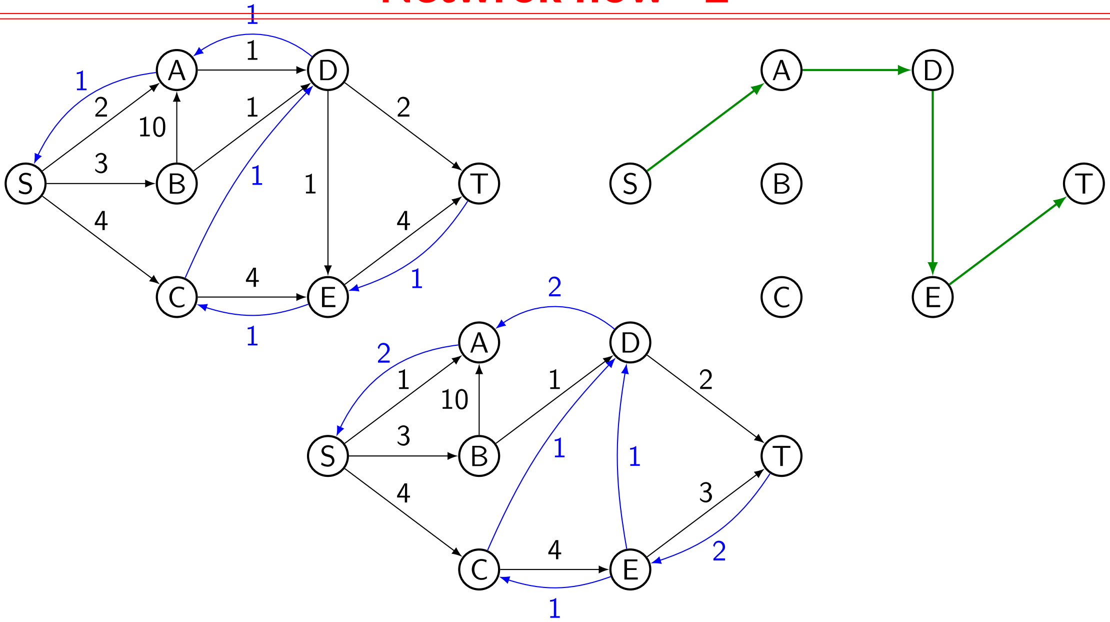


Netwrok flow - 2

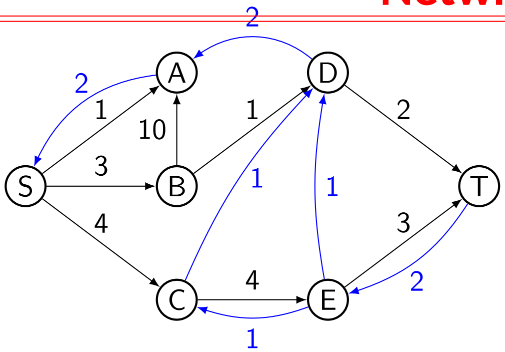




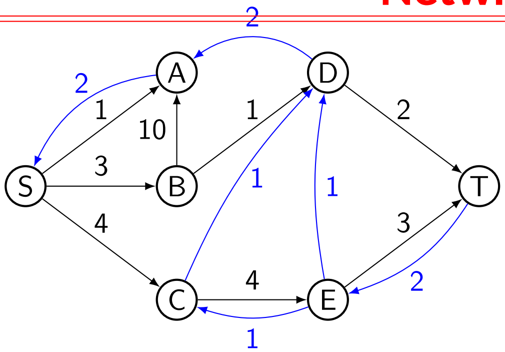
Netwrok flow - 2

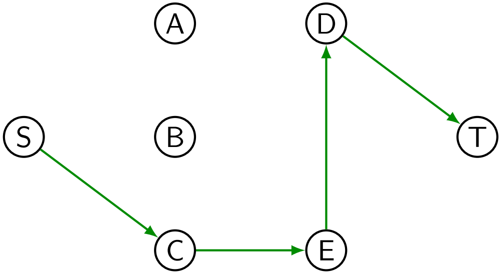


Netwrok flow - 3

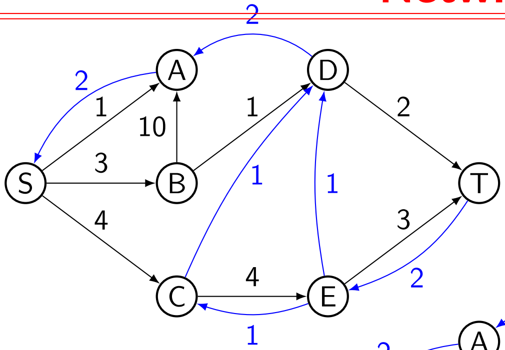


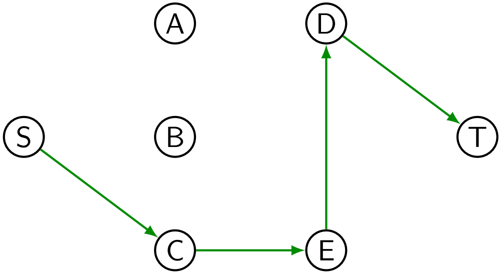
Netwrok flow - 3

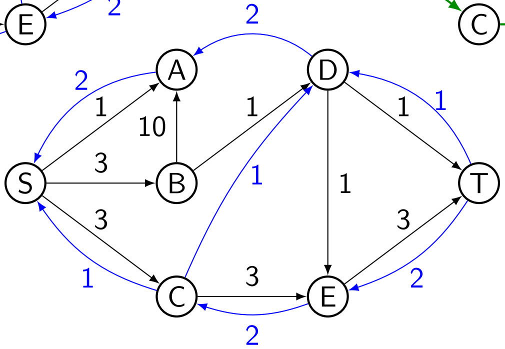




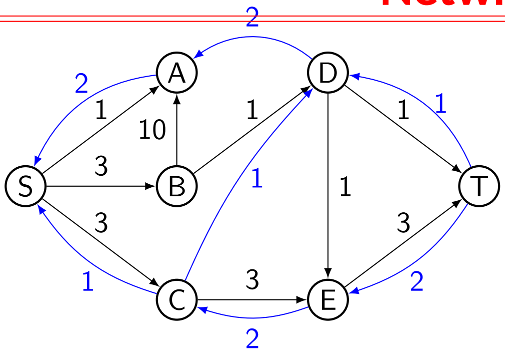
Netwrok flow - 3



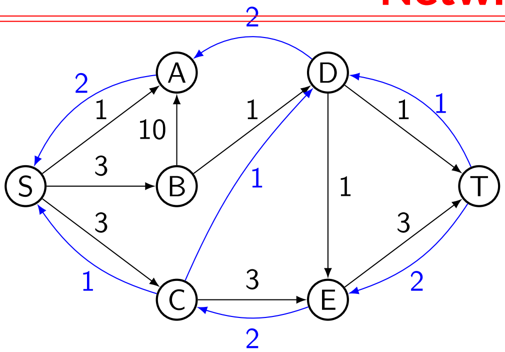


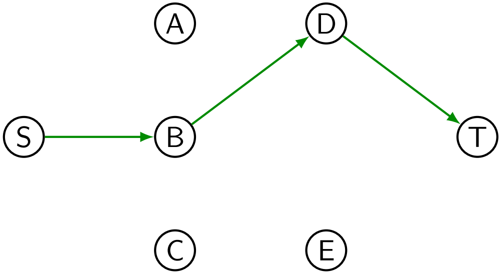


Netwrok flow - 4

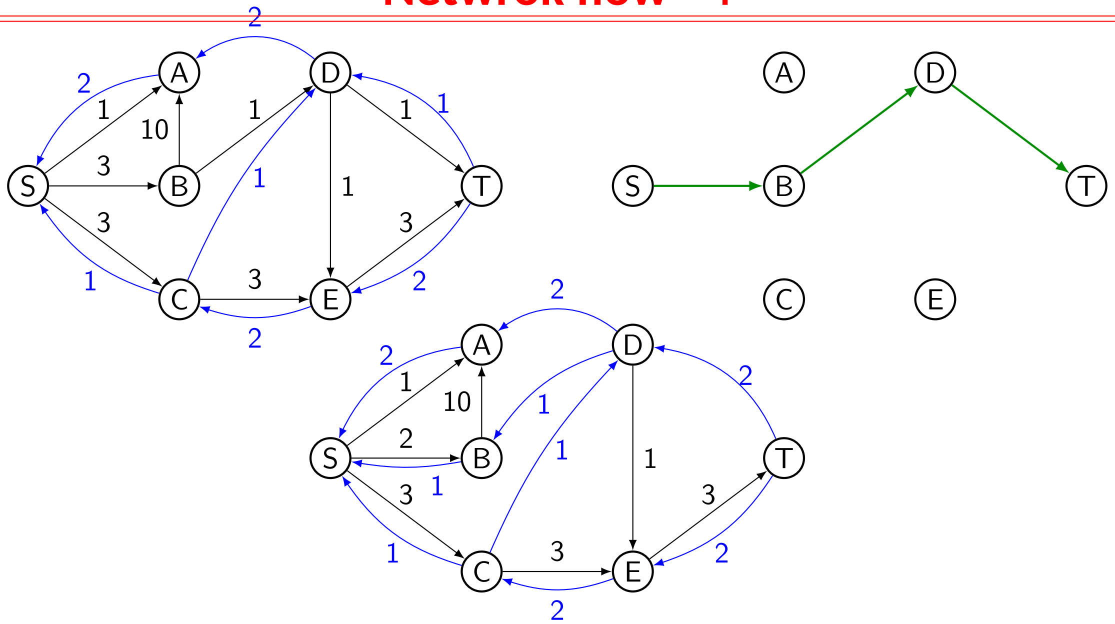


Netwrok flow - 4

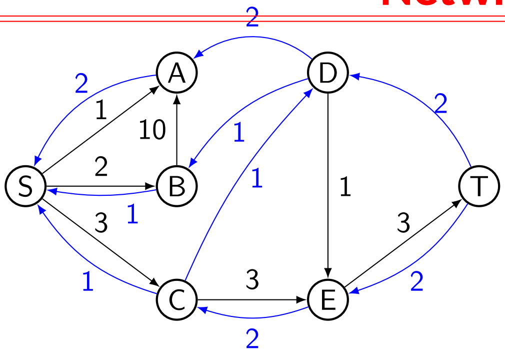




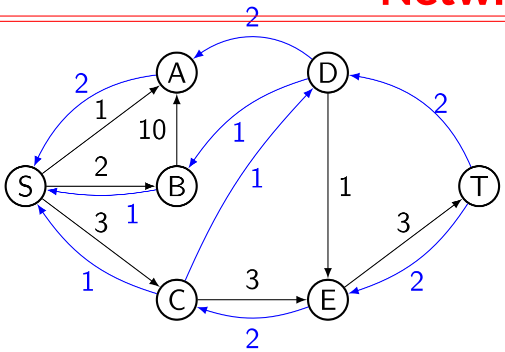
Netwrok flow - 4

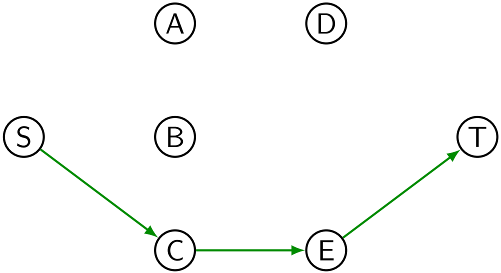


Netwrok flow - 5

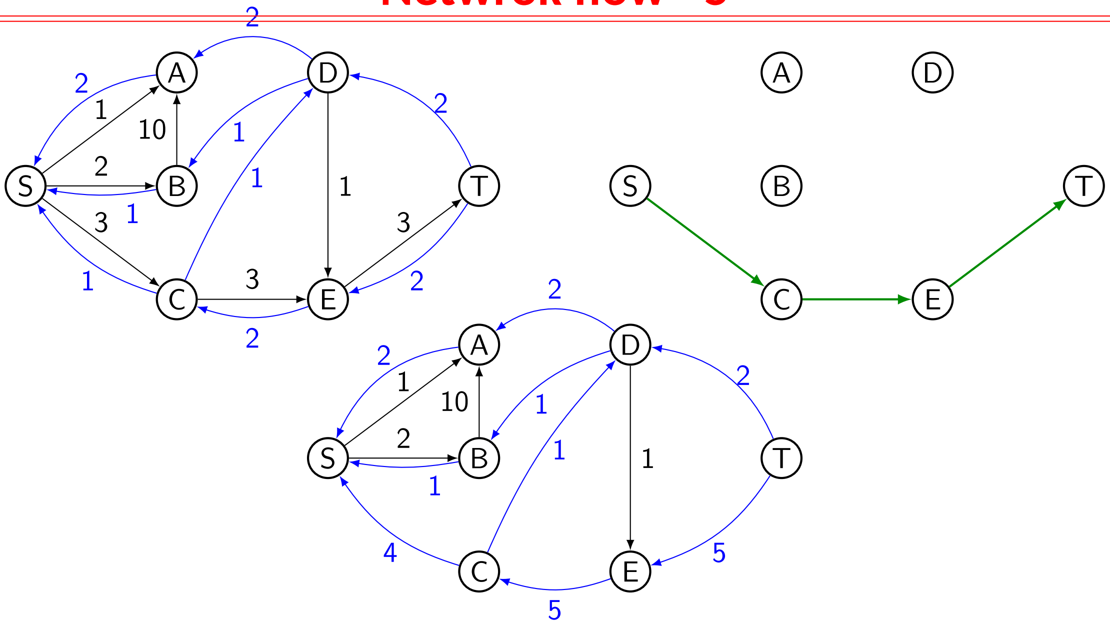


Netwrok flow - 5





Netwrok flow - 5



Augmentation

- If f is a flow in G and f' is a flow in the corresponding residual network G_f, we define $f \uparrow f'$ the augmentation of flow f by f', to be a function from $V \times V$ to $\mathbb{R}$ defined by

$$(f \uparrow f') = \begin{cases} f(u,v) + f'(u,v) - f'(v,u) & \text{if } (u,v) \in E \\ 0 & \text{otherwise} \end{cases}$$

Augmentation

- If f is a flow in G and f' is a flow in the corresponding residual network G_f, we define $f \uparrow f'$ the augmentation of flow f by f', to be a function from $V \times V$ to $\mathbb{R}$ defined by
$$(f \uparrow f') = \begin{cases} f(u, v) + f'(u, v) - f'(v, u) & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$
- Let $G = (V, E)$ be a flow network with source s and sink t and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then, $f \uparrow f' = |f| + |f'|$ holds

Proof - 1

- We have,

$$(f \uparrow f')(u, v) \quad = f(u, v) + f'(u, v) - f'(v, u)$$

Proof - 1

- We have,

$$\begin{aligned}(f \uparrow f')(u, v) &= f(u, v) + f'(u, v) - f'(v, u)\\ &\geq f(u, v) + f'(u, v) - f(u, v)\end{aligned}$$

Proof - 1

- We have,

$$
\begin{aligned}
(f \uparrow f')(u, v) &= f(u, v) + f'(u, v) - f'(v, u) \\
&\geq f(u, v) + f'(u, v) - f(u, v) \\
&= f'(u, v)
\end{aligned}
$$

Proof - 1

- We have,

$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u) \\ &\geq f(u,v) + f'(u,v) - f(u,v) \\ &= f'(u,v) \\ &\geq 0\end{aligned}$$

Proof - 1

- We have,

$$\begin{aligned}(f\uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\geq f(u,v) + f'(u,v) - f(u,v)\\ &= f'(u,v)\\ &\geq 0\end{aligned}$$

- We have

$$(f\uparrow f')(u,v) = f(u,v) + f'(u,v) - f'(v,u)$$

Proof - 1

- We have,

$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\geq f(u,v) + f'(u,v) - f(u,v)\\ &= f'(u,v)\\ &\geq 0\end{aligned}$$

- We have

$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\leq f(u,v) + f'(u,v)\end{aligned}$$

Proof - 1

- We have,

$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\geq f(u,v) + f'(u,v) - f(u,v)\\ &= f'(u,v)\\ &\geq 0\end{aligned}$$

- We have

$$\begin{aligned}(f \uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\leq f(u,v) + f'(u,v)\\ &\leq f(u,v) + c_f(u,v)\end{aligned}$$

Proof - 1

- We have,

$$\begin{aligned}(f\uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\geq f(u,v) + f'(u,v) - f(u,v)\\ &= f'(u,v)\\ &\geq 0\end{aligned}$$

- We have

$$\begin{aligned}(f\uparrow f')(u,v) &= f(u,v) + f'(u,v) - f'(v,u)\\ &\leq f(u,v) + f'(u,v)\\ &\leq f(u,v) + c_f(u,v)\\ &= f(u,v) + c(u,v) - f(u,v)\\ &= c(u,v)\end{aligned}$$

Proof - 2

- We have,

$$\sum_{v \in V} (f \uparrow f')(u, v) - \sum_{v \in V} (f \uparrow f')(v, u)$$

Proof - 2

- We have,

$$\sum_{v \in V}(f \uparrow f')(u,v) - \sum_{v \in V}(f \uparrow f')(v,u)$$
$$= \sum_{v \in V_l(u)}(f \uparrow f')(u,v) - \sum_{v \in V_e(u)}(f \uparrow f')(v,u)$$

Proof - 2

- We have,

$$\sum_{v \in V} (f \uparrow f')(u,v) - \sum_{v \in V} (f \uparrow f')(v,u)$$
$$= \sum_{v \in V_l(u)} (f \uparrow f')(u,v) - \sum_{v \in V_e(u)} (f \uparrow f')(v,u)$$
$$= \sum_{v \in V_l(u)} (f(u,v) + f'(u,v) - f'(v,u)) - \sum_{v \in V_e(u)} (f(v,u) + f'(v,u) - f'(u,v))$$

Proof - 2

- We have,

$$
\begin{aligned}
&\sum_{v \in V} (f \uparrow f')(u, v) - \sum_{v \in V} (f \uparrow f')(v, u) \\
&= \sum_{v \in V_l(u)} (f \uparrow f')(u, v) - \sum_{v \in V_e(u)} (f \uparrow f')(v, u) \\
&= \sum_{v \in V_l(u)} (f(u, v) + f'(u, v) - f'(v, u)) - \sum_{v \in V_e(u)} (f(v, u) + f'(v, u) - f'(u, v)) \\
&= \sum_{v \in V_l(u)} f(u, v) + \sum_{v \in V_l(u)} f'(u, v) - \sum_{v \in V_l(u)} f'(v, u) - \sum_{v \in V_e(u)} f(v, u) - \sum_{v \in V_e(u)} f'(v, u) + \sum_{v \in V_e(u)} f'(u, v)
\end{aligned}
$$

Proof - 2

- We have,

$$\sum_{v\in V}(f\uparrow f')(u,v) - \sum_{v\in V}(f\uparrow f')(v,u)$$

$$= \sum_{v\in V_l(u)}(f\uparrow f')(u,v) - \sum_{v\in V_e(u)}(f\uparrow f')(v,u)$$

$$= \sum_{v\in V_l(u)}(f(u,v) + f'(u,v) - f'(v,u)) - \sum_{v\in V_e(u)}(f(v,u) + f'(v,u) - f'(u,v))$$

$$= \sum_{v\in V_l(u)} f(u,v) + \sum_{v\in V_l(u)} f'(u,v) - \sum_{v\in V_l(u)} f'(v,u) - \sum_{v\in V_e(u)} f(v,u) - \sum_{v\in V_e(u)} f'(v,u) + \sum_{v\in V_e(u)} f'(u,v)$$

$$= \sum_{v\in V_l(u)} f(u,v) - \sum_{v\in V_e(u)} f(v,u) + \sum_{v\in V_l(u)} f'(u,v) + \sum_{v\in V_e(u)} f'(u,v) - \sum_{v\in V_l(u)} f'(v,u) - \sum_{v\in V_e(u)} f'(v,u)$$

Proof - 2

- We have,

$$\sum_{v \in V} (f \uparrow f')(u,v) - \sum_{v \in V} (f \uparrow f')(v,u)$$

$$= \sum_{v \in V_l(u)} (f \uparrow f')(u,v) - \sum_{v \in V_e(u)} (f \uparrow f')(v,u)$$

$$= \sum_{v \in V_l(u)} (f(u,v) + f'(u,v) - f'(v,u)) - \sum_{v \in V_e(u)} (f(v,u) + f'(v,u) - f'(u,v))$$

$$= \sum_{v \in V_l(u)} f(u,v) + \sum_{v \in V_l(u)} f'(u,v) - \sum_{v \in V_l(u)} f'(v,u) - \sum_{v \in V_e(u)} f(v,u) - \sum_{v \in V_e(u)} f'(v,u) + \sum_{v \in V_e(u)} f'(u,v)$$

$$= \sum_{v \in V_l(u)} f(u,v) - \sum_{v \in V_e(u)} f(v,u) + \sum_{v \in V_l(u)} f'(u,v) + \sum_{v \in V_e(u)} f'(u,v) - \sum_{v \in V_l(u)} f'(v,u) - \sum_{v \in V_e(u)} f'(v,u)$$

$$= \sum_{v \in V_l(u)} f(u,v) - \sum_{v \in V_e(u)} f(v,u) + \sum_{v \in V_l(u) \cup V_e(u)} f'(u,v) - \sum_{v \in V_l(u) \cup V_e(u)} f'(v,u)$$

Proof - 2

- We have,

$$\sum_{v\in V}(f\uparrow f')(u,v) - \sum_{v\in V}(f\uparrow f')(v,u)$$

$$= \sum_{v\in V_l(u)}(f\uparrow f')(u,v) - \sum_{v\in V_e(u)}(f\uparrow f')(v,u)$$

$$= \sum_{v\in V_l(u)}(f(u,v)+f'(u,v)-f'(v,u)) - \sum_{v\in V_e(u)}(f(v,u)+f'(v,u)-f'(u,v))$$

$$= \sum_{v\in V_l(u)}f(u,v) + \sum_{v\in V_l(u)}f'(u,v) - \sum_{v\in V_l(u)}f'(v,u) - \sum_{v\in V_e(u)}f(v,u) - \sum_{v\in V_e(u)}f'(v,u) + \sum_{v\in V_e(u)}f'(u,v)$$

$$= \sum_{v\in V_l(u)}f(u,v) - \sum_{v\in V_e(u)}f(v,u) + \sum_{v\in V_l(u)}f'(u,v) + \sum_{v\in V_e(u)}f'(u,v) - \sum_{v\in V_l(u)}f'(v,u) - \sum_{v\in V_e(u)}f'(v,u)$$

$$= \sum_{v\in V_l(u)}f(u,v) - \sum_{v\in V_e(u)}f(v,u) + \sum_{v\in V_l(u)\cup V_e(u)}f'(u,v) - \sum_{v\in V_l(u)\cup V_e(u)}f'(v,u)$$

- Choose $u = s$

Cut & Flow-1

- A cut (S, T) of a flow netwrok $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$

Cut & Flow-1

- A cut (S, T) of a flow netwrok $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$
- If f is flow, then the net flow $f(S, T)$ across the cut (S, T) is defined to be

$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)$$

Cut & Flow-1

- A cut (S, T) of a flow netwrok $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$
- If f is flow, then the net flow $f(S, T)$ across the cut (S, T) is defined to be
  $$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)$$
- The capacity of the cut (S, T) is $c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$

Cut & Flow-1

- A cut (S, T) of a flow netwrok $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$
- If f is flow, then the net flow $f(S, T)$ across the cut (S, T) is defined to be
$$f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)$$
- The capacity of the cut (S, T) is $c(S, T) = \sum_{u \in S} \sum_{v \in T} c(u, v)$
- A minimum cut of a netwrok is a cut whose capacity is minimum over all cuts of the network

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- **Proof**: For any vertex $u \in V - \{s, t\}$, using flow conservation condition, we can say

$$\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- **Proof**: For any vertex $u \in V - \{s, t\}$, using flow conservation condition, we can say

$$\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

- Flow at node s can be defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- **Proof**: For any vertex $u \in V - \{s, t\}$, using flow conservation condition, we can say

$$\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

- Flow at node s can be defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right)$$

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- **Proof**: For any vertex $u \in V - \{s, t\}$, using flow conservation condition, we can say

$$\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

- Flow at node s can be defined as

$$\begin{aligned} |f| &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right) \\ &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \sum_{v \in V} f(u, v) - \sum_{u \in S - \{s\}} \sum_{v \in V} f(v, u)) \end{aligned}$$

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- **Proof**: For any vertex $u \in V - \{s, t\}$, using flow conservation condition, we can say

$$\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

- Flow at node s can be defined as

$$\begin{aligned}
|f| &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right) \\
&= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S - \{s\}} \sum_{v \in V} f(u, v) - \sum_{u \in S - \{s\}} \sum_{v \in V} f(v, u)) \\
&= \sum_{v \in V} \left(f(s, v) + \sum_{u \in S - \{s\}} f(u, v) \right) - \sum_{v \in V} \left(f(v, s) + \sum_{u \in S - \{s\}} f(v, u) \right)
\end{aligned}$$

Cut & Flow-2

- Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$
- **Proof**: For any vertex $u \in V - \{s, t\}$, using flow conservation condition, we can say

$$\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) = 0$$

- Flow at node s can be defined as

$$\begin{aligned}
|f| &= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S-\{s\}} \left(\sum_{v \in V} f(u, v) - \sum_{v \in V} f(v, u) \right) \\
&= \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) + \sum_{u \in S-\{s\}} \sum_{v \in V} f(u, v) - \sum_{u \in S-\{s\}} \sum_{v \in V} f(v, u)) \\
&= \sum_{v \in V} \left(f(s, v) + \sum_{u \in S-\{s\}} f(u, v) \right) - \sum_{v \in V} \left(f(v, s) + \sum_{u \in S-\{s\}} f(v, u) \right) \\
&= \sum_{v \in V} \sum_{u \in S} f(u, v) - \sum_{v \in V} \sum_{u \in S} f(v, u)
\end{aligned}$$

Cut & Flow-3

- We have

$$|f| = \sum_{v \in V} \sum_{u \in S} f(u, v) - \sum_{v \in V} \sum_{u \in S} f(v, u)$$

Cut & Flow-3

- We have

$$|f| = \sum_{v \in V} \sum_{u \in S} f(u, v) - \sum_{v \in V} \sum_{u \in S} f(v, u) \qquad [V = S \cup T]$$

Cut & Flow-3

- We have

$$
\begin{aligned}
|f| &= \sum_{v \in V} \sum_{u \in S} f(u, v) - \sum_{v \in V} \sum_{u \in S} f(v, u) \qquad [V = S \cup T] \\
&= \sum_{v \in S} \sum_{u \in S} f(u, v) + \sum_{v \in T} \sum_{u \in S} f(u, v) - \sum_{v \in S} \sum_{u \in S} f(v, u) - \sum_{v \in T} \sum_{u \in S} f(v, u)
\end{aligned}
$$

Cut & Flow-3

- We have

$$
\begin{aligned}
|f| &= \sum_{v\in V}\sum_{u\in S} f(u,v) - \sum_{v\in V}\sum_{u\in S} f(v,u) \qquad [V = S\cup T] \\
&= \sum_{v\in S}\sum_{u\in S} f(u,v) + \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) + \left(\sum_{v\in S}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u)\right)
\end{aligned}
$$

Cut & Flow-3

- We have

$$
\begin{aligned}
|f| &= \sum_{v\in V}\sum_{u\in S} f(u,v) - \sum_{v\in V}\sum_{u\in S} f(v,u) \qquad [V = S\cup T]\\
&= \sum_{v\in S}\sum_{u\in S} f(u,v) + \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u) - \sum_{v\in T}\sum_{u\in S} f(v,u)\\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) + \left(\sum_{v\in S}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u)\right)\\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u)
\end{aligned}
$$

Cut & Flow-3

- We have

$$
\begin{aligned}
|f| &= \sum_{v\in V}\sum_{u\in S} f(u,v) - \sum_{v\in V}\sum_{u\in S} f(v,u) \qquad [V = S \cup T] \\
&= \sum_{v\in S}\sum_{u\in S} f(u,v) + \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) + \left(\sum_{v\in S}\sum_{u\in S} f(u,v) - \sum_{v\in S}\sum_{u\in S} f(v,u)\right) \\
&= \sum_{v\in T}\sum_{u\in S} f(u,v) - \sum_{v\in T}\sum_{u\in S} f(v,u) \\
&= f(S,T)
\end{aligned}
$$

Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G

Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G
- Proof:

$$|f| = f(S, T)$$

Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G
- Proof:

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u)
\end{aligned}
$$

Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G
- Proof:

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \\
&\leq \sum_{u \in S} \sum_{v \in T} f(u, v)
\end{aligned}
$$

Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G
- Proof:

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \\
&\leq \sum_{u \in S} \sum_{v \in T} f(u, v) \\
&\leq \sum_{u \in S} \sum_{v \in T} c(u, v)
\end{aligned}
$$

Cut & Capacity

- The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G
- Proof:

$$
\begin{aligned}
|f| &= f(S, T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u, v) - \sum_{u \in S} \sum_{v \in T} f(v, u) \\
&\leq \sum_{u \in S} \sum_{v \in T} f(u, v) \\
&\leq \sum_{u \in S} \sum_{v \in T} c(u, v) \\
&= c(S, T)
\end{aligned}
$$

Max-flow min-cut

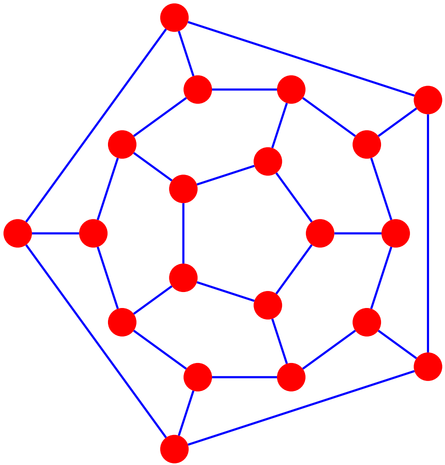
- If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:
  - f is a maximum flow in G
  - The residual network G_f contains no augmenting paths
  - $|f| = c(S, T)$ for some cut (S, T) of G

Edmonds-Karp algorithm

- Augmenting path with fewest edges needs to be chosen
- Breadth-first search can be used to find augmenting path in the residual network
- Time complexity becomes $O(VE^2)$

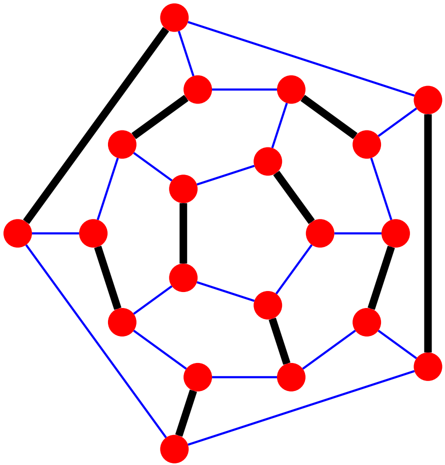
Matching

- Given an undirected graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a matching if each node of the graph appears *at most* one edge of M.



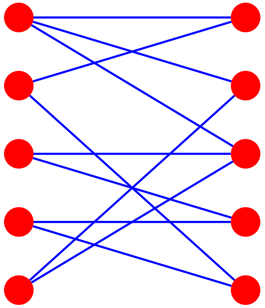
Matching

- Given an undirected graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a matching if each node of the graph appears *at most* one edge of M.



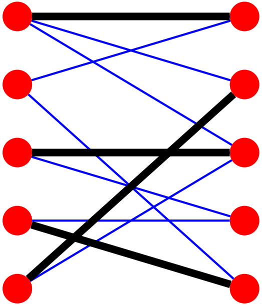
Bipartite matching

- A graph is bipartite if the nodes can be partitioned into two subsets X and Y such that every edge connects a node in X to a node in Y
- Given a bipartite graph $G = (X \cup Y, E)$, find a matching (M) that has the maximum cardinality ie., $|M|$ is maximum.



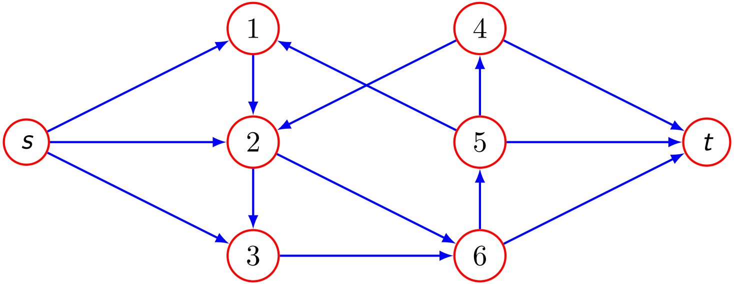
Bipartite matching

- A graph is bipartite if the nodes can be partitioned into two subsets X and Y such that every edge connects a node in X to a node in Y
- Given a bipartite graph $G = (X \cup Y, E)$, find a matching (M) that has the maximum cardinality ie., $|M|$ is maximum.



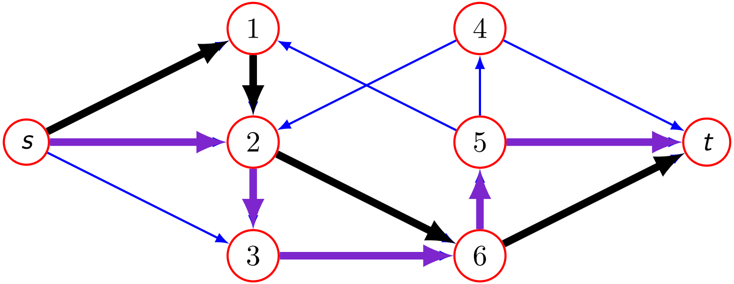
Edge-disjoint paths

- Two paths are edge-disjoint if they have no common edge. Given a directed graph $G = (V, E)$ and two nodes s and t, find the maximum number of edge-disjoint $s \rightsquigarrow t$ paths.



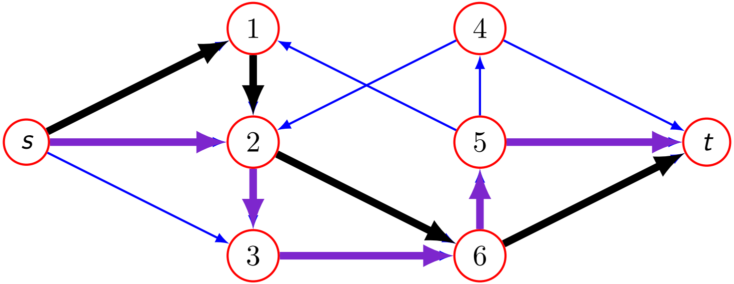
Edge-disjoint paths

- Two paths are edge-disjoint if they have no common edge. Given a directed graph $G = (V, E)$ and two nodes s and t, find the maximum number of edge-disjoint $s \rightsquigarrow t$ paths.



Network connectivity

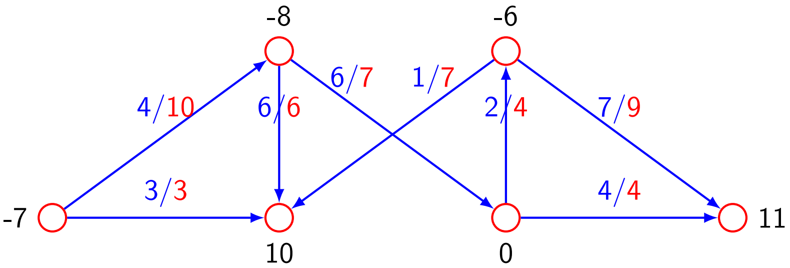
- Given a digraph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.



Circulation with demands

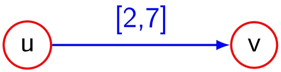
- Given a directed graph $V = (G, E)$ with non-negative edge capacities $c(e)$ and node supply and demands $d(v)$, a circulation is a function that satisfies
  - For each $e \in E$: $0 \leq f(e) \leq c(e)$ ($f(.)$ – flow along edge e)
  - For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$

  Does a circulation exist?
- $d(v) > 0$ - demand, $d(v) < 0$ - supply, $d(v) = 0$ - transshipment node



Circulation with lower bounds

- The problem is the same as previous one except that each edge has some lower bound on the flow. Hence, capacity along an edge will be specified as $[c_{lb}(u,v), c_{ub}(u,v)]$. What modifications are to be made in the graph to apply previous strategy?



Survey design

- Design a survey asking n_1 consumers about n_2 products that meets the following requirements, if possible.
  - Consumer i can survey about product j if they own it
  - Consumer i can be asked between c_i and c'_i questions
  - Ask between p_j and p'_j consumers about product j

Thank you!