CS514: Design and Analysis of Algorithms

Intractability

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- Euler tour vs Hamiltonian cycle
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- 2-SAT vs 3-SAT
  - 2-SAT: $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3)$
  - 3-SAT: $(x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

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- Fractional vs 0-1 knapsack

Time complexity

Time complexity function	Size n					
	10	20	30	40	50	60
n	.00001 second	.00002 second	.00003 second	.00004 second	.00005 second	.00006 second
n^2	.0001 second	.0004 second	.0009 second	.0016 second	.0025 second	.0036 second
n^3	.001 second	.008 second	.027 second	.064 second	.125 second	.216 second
n^5	.1 second	3.2 seconds	24.3 seconds	1.7 minutes	5.2 minutes	13.0 minutes
2^n	.001 second	1.0 second	17.9 minutes	12.7 days	35.7 years	366 centuries
3^n	.059 second	58 minutes	6.5 years	3855 centuries	2×10^8 centuries	1.3×10^{13} centuries

Image source: Computers and Intractability

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- It is obvious that P $\subseteq$ NP. However, the famous open question is whether P is a proper subset of NP
- If any NP-complete problem can be solved in polynomial time, then every problem in NP has a polynomial-time algorithm

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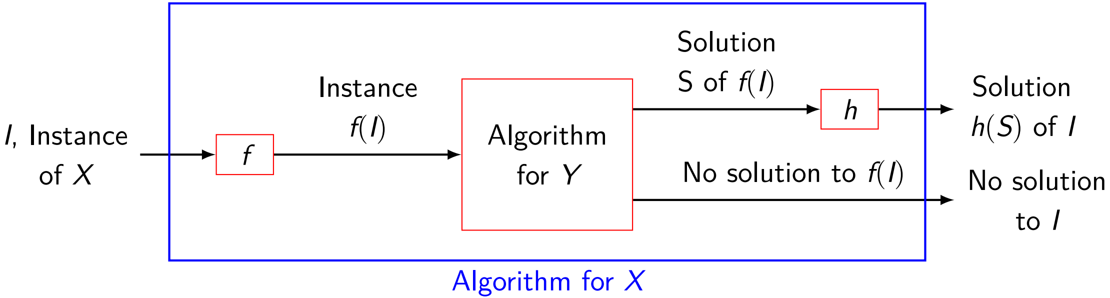
- Which problem is harder?
- Can an optimization problem be converted as decision problem?

Reduction

- $X \leq_p Y$
- Problem X polynomial-time reduces to problem Y if arbitrary instances of problem X can be solved using:
  - Polynomial number of standard computational steps (f, h), plus
  - Polynomial number of calls to oracle that solves problem Y



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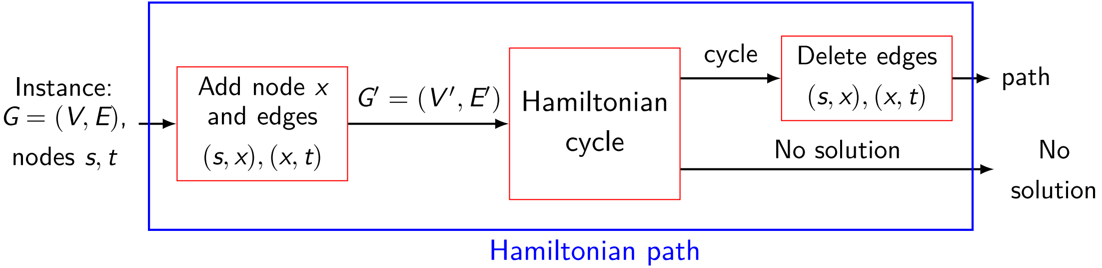
- If $X \leq_p Y$ and Y can be solved in exponential time, then X – ??

Hamiltonian path $\to$ Hamiltonian cycle

- Hamiltonian cycle: given a graph, is there a cycle that passes through each vertex exactly once?
- Hamiltonian path(s, t): given a graph, is there a path between s and t that passes through each vertex exactly once?



Abstract problem

- An abstract problem Q is defined to be binary relation on a set I of problem instances and a set S of problem solution
  - For shortest-path – problem instance consists of a graph and two vertices, $I = \langle G, u, v \rangle$
  - A solution is sequence of vertices or null if it does not exist
- For NP-Completeness, we are primarily interested in decision problems
- For shortest-path, decision problem can be represented as $I = \langle G, u, v, k \rangle$
  - Given a graph and two vertices, does there exist a path with at most k edges?
- An optimization problem can be converted to decision problem

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Encoding scheme	string	length
Vertex and edge list	$v_1 v_2 v_3 v_4 (v_1 v_2)(v_2 v_3)$	36
Neighbor list	$(v_2)(v_1 v_3)(v_2)()$	24
Adjacency matrix rows	0100/1010/0100/0000	19

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- A concrete problem is polynomial-time solvable if there exist an algorithm to solve it in $O(n^k)$ time for some constant k

Encoding

- We say a function $f\colon \{0,1\}^* \to \{0,1\}^*$ is polynomial-time computable if there exists a polynomial-time algorithm A that given any input $x \in \{0,1\}^*$, produces as output $f(x)$
- We say that two encodings e_1 and e_2 are polynomially related if there exist two polynomial-time computable function f_{12} and f_{21} such that for any $i \in I$, we have $f_{12}(e_1(i)) = e_2(i)$ and $f_{21}(e_2(i)) = e_1(i)$

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- Let Q be an abstract decision problem on an instance set I, and let e_1 and e_2 be polynomially related encodings on I. Then, $e_1(Q) \in \mathrm{P}$ if and only if $e_2(Q) \in \mathrm{P}$.

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- It leaves the question of whether P $=$ NP

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- If any NP-complete problem is polynomial-time solvable, then $\mathrm{P} = \mathrm{NP}$. Equivalently, if any problem in NP is not polynomial-time solvable, then no NP-complete problem is polynomial-time solvable.

Circuit-SAT

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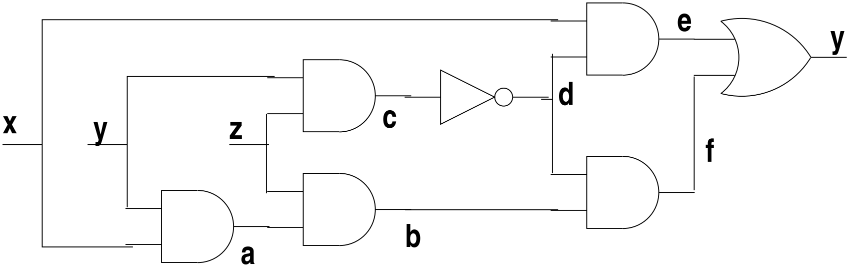

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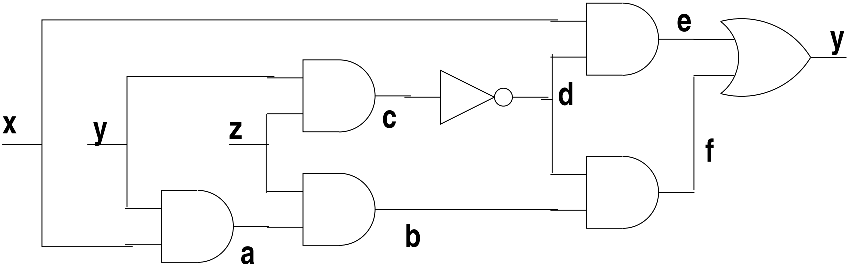

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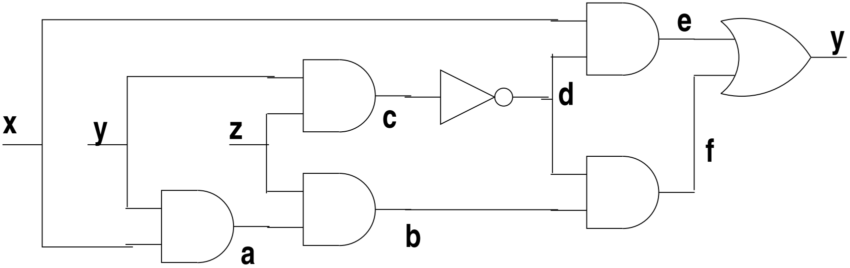
- Circuit-SAT is also NP-Hard (see detailed proof in the book)



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- Prove $L \in \text{NP}$
- Prove that L is NP-hard:
  - Select a known NP-complete language L'
  - Describe an algorithm that computes a function f mapping every instance $x \in \{0,1\}^*$ of L' to an instance of $f(x)$ of L
  - Prove that the function f satisfies $x \in L'$ if and only if $f(x) \in L$ for all $x \in \{0,1\}^*$
  - Prove that the algorithm computing f runs in polynomial time

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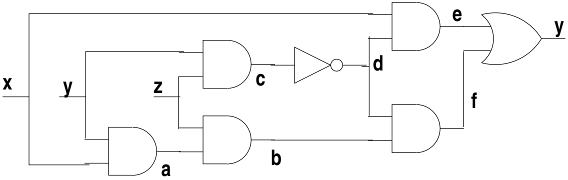

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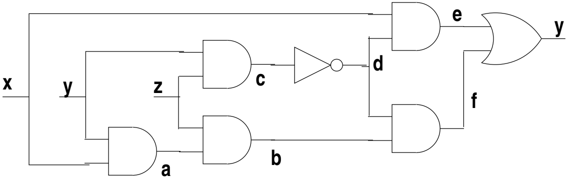
$$\begin{aligned}\phi = \quad & y \wedge (y \leftrightarrow (e \vee f)) \wedge (e \leftrightarrow (x \wedge d)) \wedge \\ & (f \leftrightarrow (d \wedge b)) \wedge (d \leftrightarrow \neg c) \wedge \\ & (b \leftrightarrow (z \wedge a)) \wedge (c \leftrightarrow (z \wedge y)) \wedge \\ & (a \leftrightarrow (x \wedge y))\end{aligned}$$



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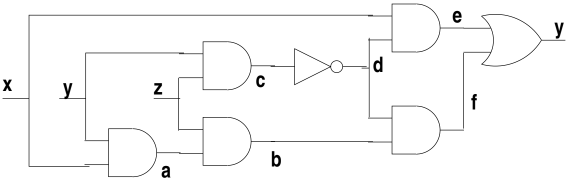

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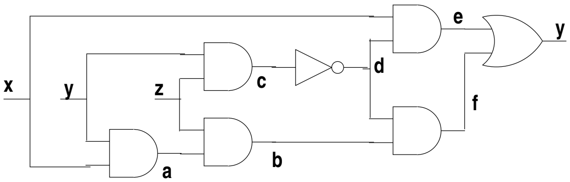

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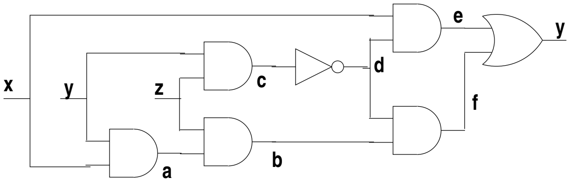

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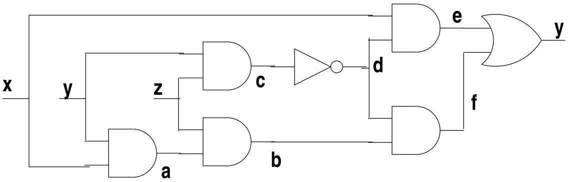

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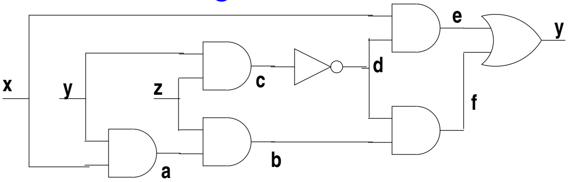
- If some assignment causes ϕ to evaluate to 1, we can assign values to different wires and it will evaluate to 1 for C



Circuit-SAT $\leq_P$ CNF-SAT

- Literal - variable in boolean formula, x_1 or $\neg x_1$
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- Circuit-SAT can be converted to CNF-SAT in polynomial time using above transformations
- It can be shown Circuit-SAT has a solution if and only if CNF-SAT has a solution

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  Set $z_1, \ldots, z_{i-2}$ to `true` and rest to `false`

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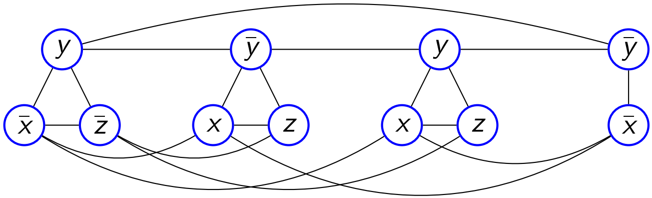

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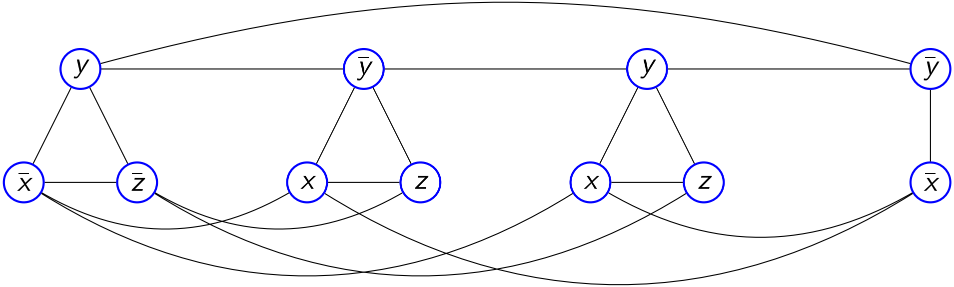

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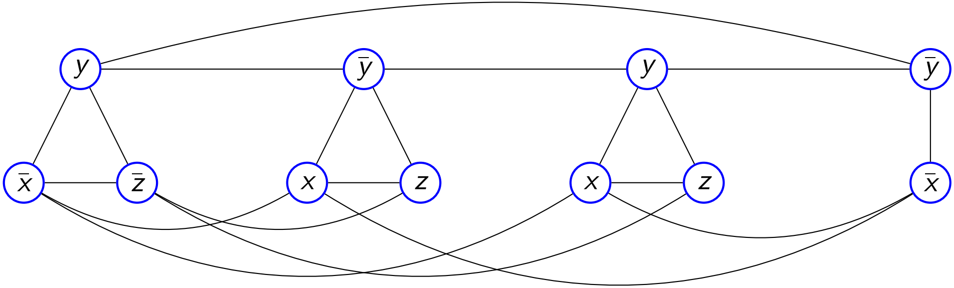
- If I has a truth assignment then G has independent set of size k



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- The remaining nodes $V - S$ must form an independent set!

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- A vertex cover of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $(u, v) \in E$ then $u \in V'$ or $v \in V'$ or both. Does graph G has a vertex cover of size k?
- Let a set of nodes S be the vertex cover of G, that is S touches every edge in E
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- Thus to solve, an instance of (G, k) of independent-set, we simply look for a vertex cover of G with $V - k$ nodes
- If a vertex cover exists, then all nodes not in VC set form IS
- If no such vertex cover exists, G cannot have an independent set of size k

Clique $\in$ NPC

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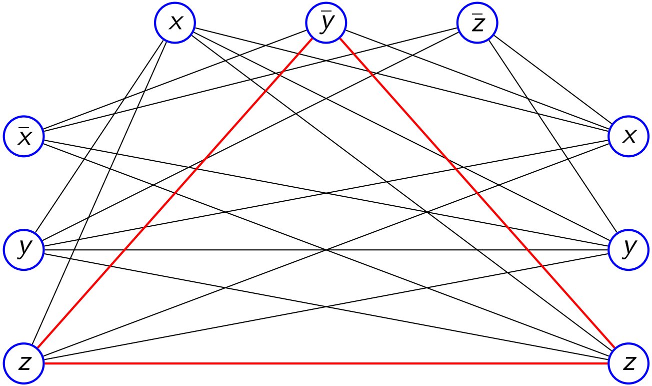
- We choose X, CNF-SAT instance, that has k number of clauses $(C_1, \ldots, C_k)$, where each clause has exactly 3 literals
- Graph construction: $G = (V, E)$
  - For each clause $C_r = (l_1^r \vee l_2^r \vee l_3^r)$ in X create three vertices v_1^r, v_2^r, v_3^r into V
  - Add edge (v_i^r, v_j^s) into E if both of the following hold
    - v_i^r and v_j^s are in different triples, that is $r \neq s$, and
    - their corresponding literals are consistent, that is l_i^r is not the negation of l_j^s

Clique $\in$ NPC

- Consider a 3SAT instance as $X = C_1 \wedge C_2 \wedge C_3 = (\bar{x} \vee y \vee z)(x \vee \bar{y} \vee \bar{z})(x \vee y \vee z)$



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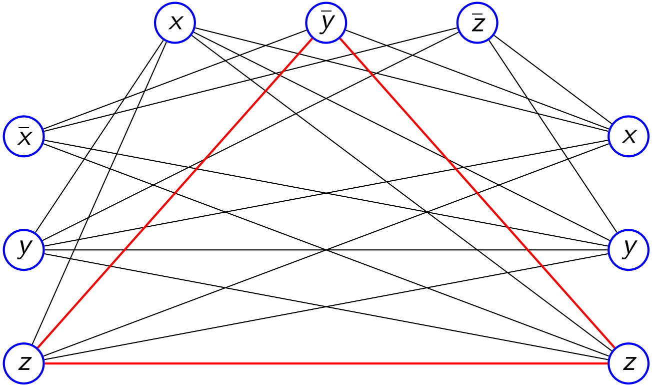

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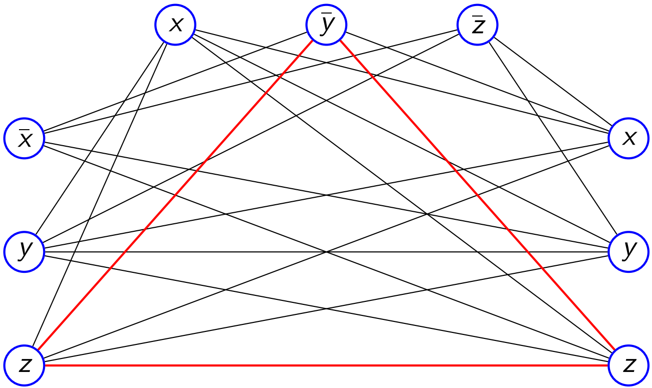
- Suppose X has a satisfying assignment: what can we claim?
- Suppose G contains a clique of size k: what can be claimed?



Thank you!