CS514: Design and Analysis of Algorithms

Recursion

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Integer Multiplication - 1

- Input: Two non-negative integers, x, y
- Output: The product, $x \times y$

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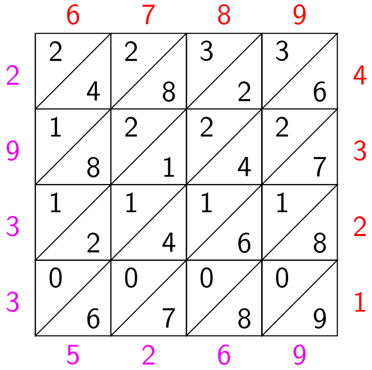

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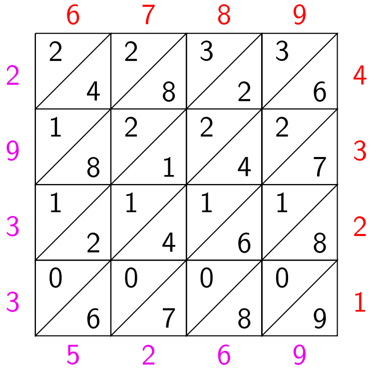

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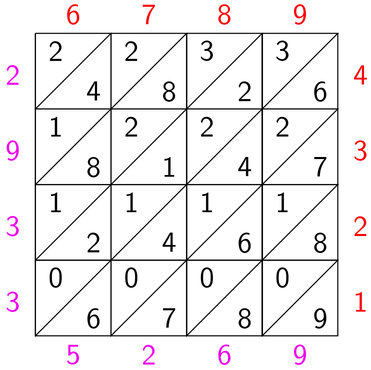

- Number of operations performed?
- Time complexity?

Integer Multiplication - 2

- mult1(x,y,n):
  1. if n==0 return 0;
  2. $x' =$ x[n,2]; $x_1 =$ x[1];
  3. m=mult1(x', y, n-1);
  4. return $10 \times m + x_1 \times y$

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- mult2(x,y,n)

1. if y==0 return 0
2. z = mult2(x, $\lfloor y/2 \rfloor$)
3. if y is even then return $2 \times m$
4. else return $2 \times m + x$

Integer Multiplication - 3

- mult3(x,y,n):
  1. if n==1 return $x * y$;
  2. x_l, x_r = left and right half of x
  3. y_l, y_r = left and right half of y
  4. m_1=mult3$(x_l, y_l, n/2)$;
  5. m_2=mult3$(x_r, y_r, n/2)$;
  6. m_3=mult3$(x_l, y_r, n/2)$;
  7. m_4=mult3$(x_r, y_l, n/2)$;
  8. return $2^n * m_1 + 2^{n/2} * (m_3 + m_4) + m_2$

Integer Multiplication - 4

- mult4(x,y,n):

1. if n==1 return $x * y$;
2. x_l, x_r = left and right half of x
3. y_l, y_r = left and right half of y
4. m_1=mult4$(x_l, y_l, n/2)$;
5. m_2=mult4$(x_r, y_r, n/2)$;
6. m_3=mult4$(x_l + x_r, y_l + y_r, n/2)$;
7. return $2^n * m_1 + 2^{n/2} * (m_3 - m_1 - m_2) + m_2$

Asymptotic analysis

- Run time depends on - machine specification (instruction set, memory, cache, cpu speed, etc.), algorithms, input size

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- Need a framework for machine independent comparison
- Suppress constant factors and lower order terms

Asymptotic notations

- Big-O
  - $T(n) = O(f(n))$ if and only if $T(n)$ is eventually bounded above by a constant multiple of $f(n)$

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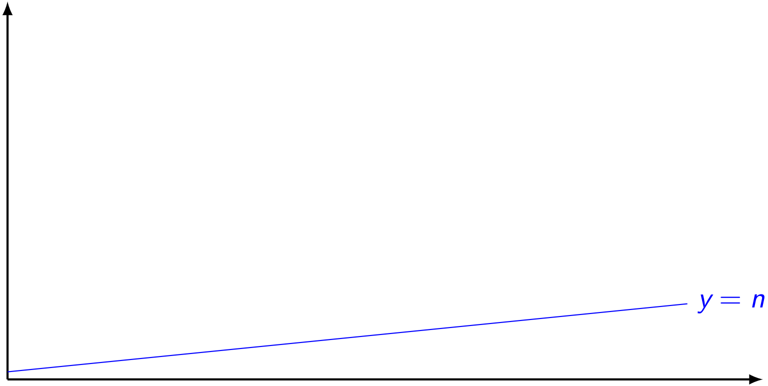
Growth of functions



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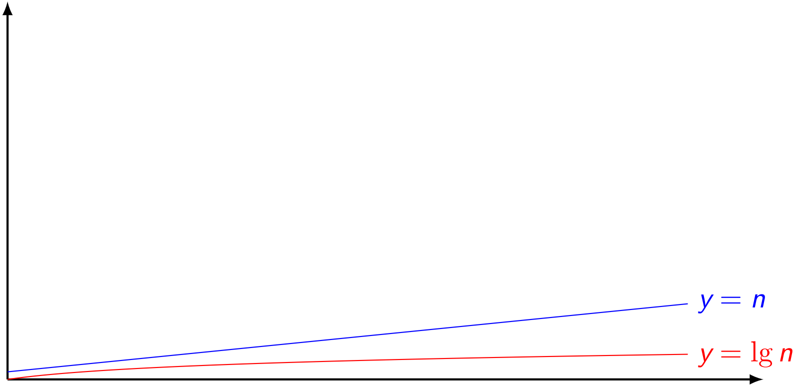

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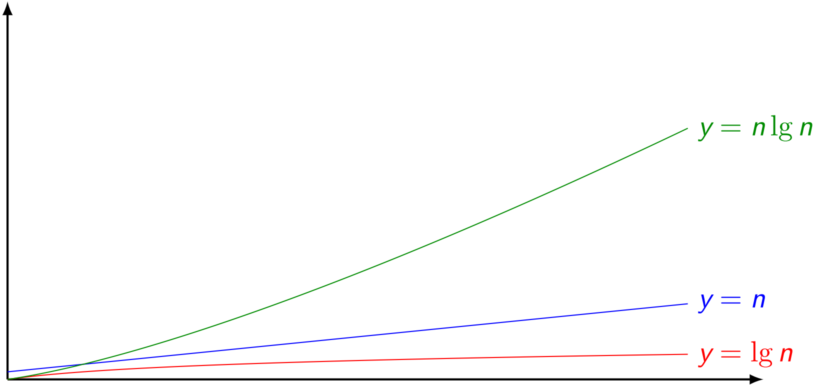

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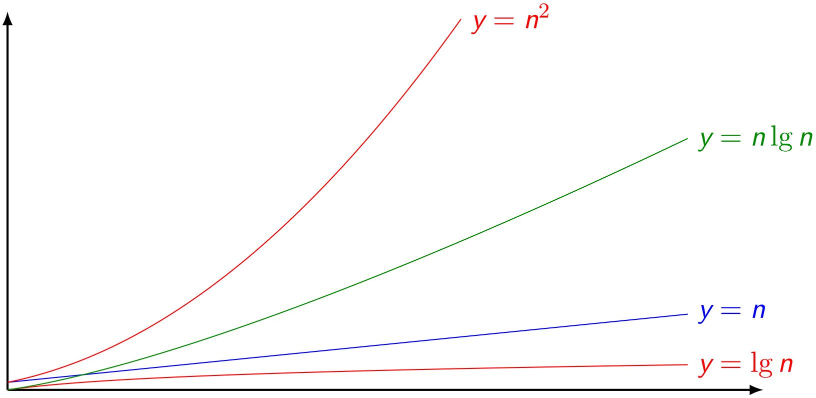

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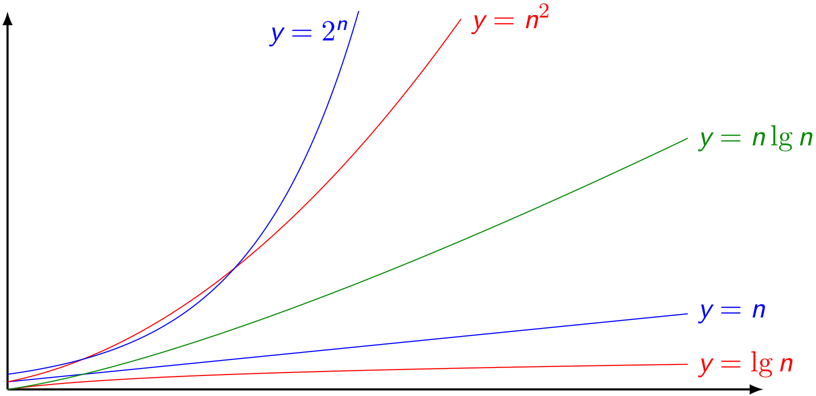

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  - Example: $n, 5n, 10n + \lg n, n + 10000\sqrt{(}n), n^2 + 10000 \lg n, n^{\lg 4}, n^{1.99}, 2^n, n^2 \lg n, n^{2.01} - \Omega(n), \Omega(n^2)?$

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  - $T(n) = \frac{1}{2}n^2 + 3n$ — $O(n), \Omega(n), \Theta(n^2), O(n^3)$?

Recursion Example

- Tower of Hanoi
- Matrix multiplication

Solving recurrences

- Substitution method: Guess the solution and prove by mathematical induction
  - $T(n) = 2T(\lfloor n/2 \rfloor) + O(n)$ — Try with $O(n^2)$, $O(n \lg n)$, $O(n)$

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- Recursion tree
  - $T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$

Master theorem

- If $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ for some constants $a > 0$, $b > 1$, and $d \geq 0$ then

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

Master theorem: Examples

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- $T(n) = 2T(n/4) + n^{0.51}$

Excercise

- Solve the recurrence relation:
  - $T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \lg n$

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- Solve the recurrence relation:
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  - $T(n) = T(n/3) + T(2n/3) + n$

Thank you!