GOVERNING EQUATIONS - III.

Passive MEMS structures: Inkjet nozzles, etc

Active MEMS structures: Micropumps, etc

Microfluidics

Non-linear PDEs
(Navier-Stokes eqn.).

GOVERNING EQUATIONS

Conservation of mass

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial u_i}{\partial x_i} = 0. \qquad \text{(I)}$$

velocities

Define strain-rate tensor,

$$\frac{\partial \epsilon_{ij}}{\partial t} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad \text{(II)}$$

(Note: $u_i \rightarrow$ velocities).

Stress in the fluid is described by σ_{ij}, s.t.,

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\,\dot{\epsilon}_{ij} + \lambda\,\dot{\epsilon}_{kk}\,\delta_{ij}. \qquad \text{(III)}$$

$p \Rightarrow$ pressure in the fluid.
$\mu \Rightarrow$ dynamic fluid viscosity.
$\lambda \Rightarrow$ second viscosity coefficient.

Conservation of momentum $\Rightarrow$

$$\underbrace{\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j}}_{\text{mass} \times \text{acceleration}} = \underbrace{\rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j}}_{\text{Forces}} . \quad \text{(IV)}$$

for an arbitrary volume.

Using (III) in (IV),

$$\Rightarrow \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\mu \dot{\epsilon}_{ij} + \lambda \dot{\epsilon}_{kk} \delta_{ij}\right). \quad \text{(V)}$$

Eqn. (I) & (V) are called the Navier-Stokes equations.

In vector form, they are,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \quad \text{— (VI)}$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho (\vec{u} \cdot \nabla)\vec{u} + \nabla p - \mu \nabla^2 \vec{u} - (\lambda + \mu) \nabla (\nabla \cdot \vec{u}) = \vec{f} . \quad \text{(VII)}$$

System of four coupled nonlinear partial differential equations for a viscous compressible fluid.

Special case:

Incompressible NS equation for a flow with characteristic velocity U, in a spatial region of characteristic length l.

$$\vec{u}^* = \frac{\vec{u}}{U}, \quad x^* = \frac{x}{l}, \quad y^* = \frac{y}{l}, \quad z^* = \frac{z}{l},$$

$$t^* = \frac{Ut}{l}, \quad p^* = \frac{p - p_\infty}{\rho U^2}.$$

$\Downarrow$ dimensionless system

$$\nabla \cdot \vec{u}^* = 0$$

$$\frac{\partial \vec{u}^*}{\partial t^*} + (\vec{u}^* \cdot \nabla)\vec{u}^* = -\nabla p^* + \frac{1}{Re}\nabla^2 \vec{u}^*.$$

$$Re = \frac{Ul}{\nu}.$$

$Re = 0 \Rightarrow$ Stokes equations.

$\Downarrow$

Of particular interest to Microfluidics.

Boundary conditions

~~At solif~~ solid-fluid boundary:

No-penetration B.C. $\vec{u}\cdot\hat{n} = 0$.

No-slip B.C. $\vec{u}\times\hat{n} = 0$.

Together $\Rightarrow$ $\vec{u} = 0$ at boundary.

Electrodynamics. (brief recap).

$$\left.\begin{aligned} \nabla\times\vec{E} &= \\ \nabla\times\vec{B} &= \\ \nabla\cdot\vec{E} &= \\ \nabla\cdot\vec{B} &= \end{aligned}\right\} \text{maxwell's equations.}$$

- $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$.

- $\dfrac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$.

$D_i = \epsilon_{ij} E_j$.

$B_i = \mu_{ij} H_j$.

B.C.

$(\vec{D}_2 - \vec{D}_1)\cdot\hat{n} = \sigma$.

$(\vec{B}_2 - \vec{B}_1)\cdot\hat{n} = 0$.

$(\vec{E}_2 - \vec{E}_1)\times\hat{n} = 0$.

$(\vec{H}_2 - \vec{H}_1)\times\hat{n} = -\vec{J}_s$.

Note :

1. Five unknown functions viz., p, ρ and three components of $\vec{u}$ (velocity vector field).

2. Conservation of energy $\Rightarrow$ 1 more equation & 1 unknown function T (temperature).

3. Introduce equation of state which relates ρ, p & T.

$$\Updownarrow$$

Define the problem completely.

Analytic solution:

Elusive

(otelbaev)

Mukhtarbay Otelbayev of Eurasian National University in Astana, Kazakhstan proposed a solution in 2013.

(One flaw detected & community is after it.)

CASE I: Incompressible fluid.

ρ is a constant.

$\therefore$ Navier-Stokes equations yield,

$$\text{VIII.} \quad \left\{ \begin{array}{l} \nabla \cdot \vec{u} = 0 \\ \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\dfrac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}. \end{array} \right.$$

where, $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity.

(system of four equations in four unknowns).

CASE II: Inviscid & incompressible fluid

$\nu = 0$, ρ = constant.

$\therefore$ Navier-Stokes equations reduce to:

(Euler equation)

$$\text{IX.} \quad \left\{ \begin{array}{l} \nabla \cdot \vec{u} = 0 \\ \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\dfrac{1}{\rho}\nabla p. \end{array} \right.$$