

Variable mass situation

PH101 (ADT)

L9

①

(I.) Condensation of a water droplet

A dust particle of negligible mass at $t=0$ begins to fall under the influence of gravity through saturated water vapour. The vapour condenses with constant rate λ (gm-cm^{-1}) on the dust particle & forms a water droplet of steadily increasing mass. [neglect friction & collisions].

The external force acting on droplet is gravity:

$$F_g = mg.$$

$$\Rightarrow mg = \frac{dm}{dt} v + m \frac{dv}{dt}.$$

$$\text{But } \frac{dm}{dt} = \frac{dm}{dx} \cdot \frac{dx}{dt} = \lambda v.$$

$$\therefore mg = \lambda v^2 + m \frac{dv}{dt}.$$

$$\Rightarrow a = \frac{dv}{dt} = \left(\frac{mg - \lambda v^2}{m} \right)$$

$$\text{At } t=0, x=0 \text{ \& } m=0.$$

$$\Rightarrow m = \lambda x.$$

$$\therefore \boxed{a = g - \frac{v^2}{x}}.$$

$$\Rightarrow x \cdot \frac{d^2x}{dt^2} + \left(\frac{dx}{dt} \right)^2 - gx = 0.$$

To solve this nonlinear differential equation, try ansatz $x = At^n$.

$$\Rightarrow (At^n) n(n-1)A t^{n-2} + (nAt^{n-1})^2 - gAt^n = 0.$$

$$\text{i.e., } \boxed{n(n-1)A^2 t^{2n-2} + n^2 A^2 t^{2n-2} - gAt^n = 0}$$

Demand $2n-2 = n$.

$$\Rightarrow n=2.$$

Substituting in the boxed eq. above,

$$\Rightarrow 2A^2 t^2 + 4A^2 t^2 - gAt^2 = 0.$$

$$\therefore 6A^2 = gA.$$

$$\Rightarrow A = \frac{g}{6}.$$

$$\therefore \boxed{x = \frac{g}{6} t^2}$$

$$\therefore \boxed{a = \frac{d^2 x}{dt^2} = \frac{g}{3}}$$

\therefore Accⁿ of the droplet is constant & independent of x , & equals $\frac{g}{3}$.

Other examples

- Rocket motion
- People travelling on an escalator
- sand on a conveyor belt.
- Leaky bucket
- Chain released on a weight measuring scale.

(II.) Chain falling on a weighing scale

$$\text{Length} = L.$$

$$\text{mass per unit length} = \sigma.$$

Method - I

Let y = height of top of the chain.

Let F be the desired force applied by the scale.

The net force on the entire chain is

$$F - (\sigma L)g.$$

Momentum of entire chain

$$= (\sigma y) \dot{y}. \quad \left(\begin{array}{l} \text{only} \\ \text{moving part} \\ \text{contributes} \end{array} \right)$$

$$\therefore F - (\sigma L)g = \sigma y \ddot{y} + \sigma \dot{y}^2. \quad \text{--- (I)}$$

Part of chain which is still above the scale is in "free fall", i.e., $\ddot{y} = -g$.

Also conservation of energy

$$\Rightarrow \sigma(L-y)gy = \frac{1}{2} \sigma y \dot{y}^2$$

$$\Rightarrow y = \sqrt{2g(L-y)}$$

(4)

putting in eq. (I),

$$\Rightarrow F = \sigma Lg - \sigma yg + 2\sigma(L-y)g.$$

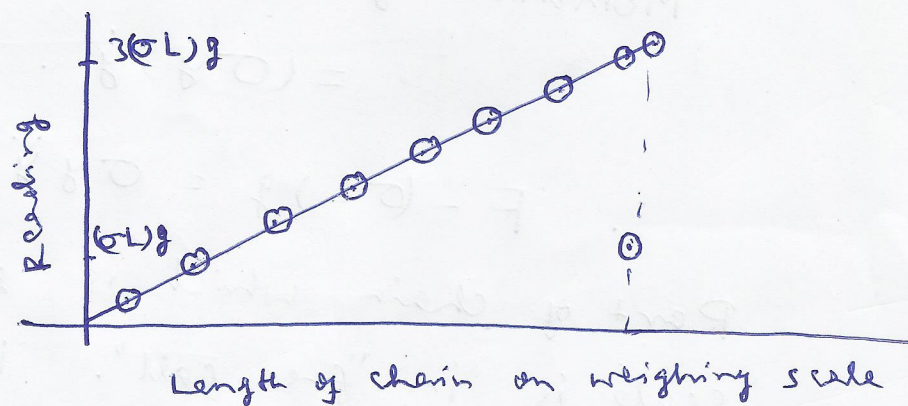
$$\text{i.e., } F = 3\sigma(L-y)g.$$

Three times the weight of the chain already on the scale.

Check! At $y = L$, $F = 0$.
(Entire chain hanging)

At $y = 0$, $F = 3(\sigma L)g$.
(Just before the last bit falls on the scale)

NOTE: Once the chain is completely on the scale, reading drops down to $(\sigma L)g$.



(III.) Rockets(a.) Free space

At $t=0$, the ignition happens & the fuel products are ejected backwards with speed u relative to the rocket.

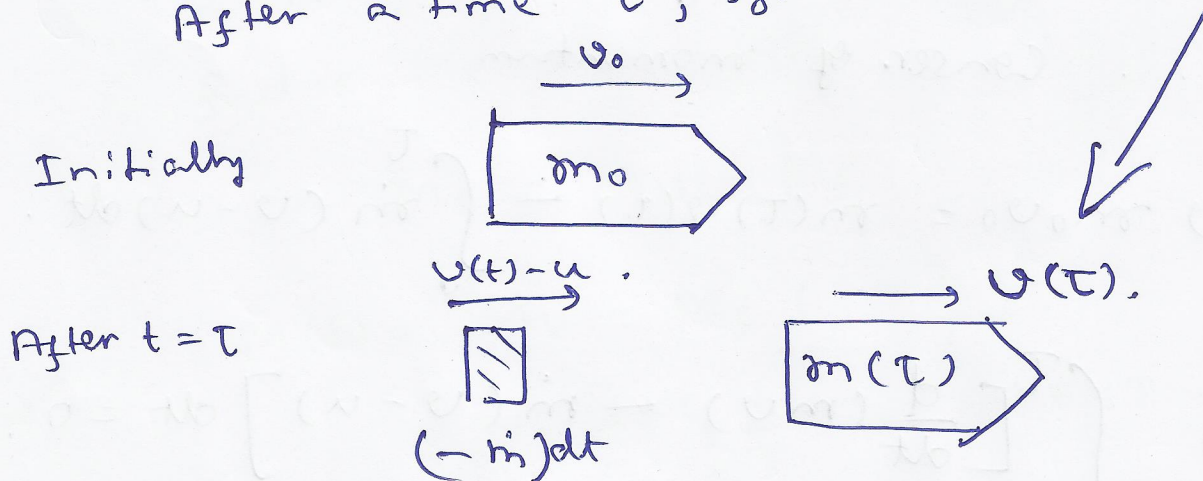
The fuel burn continues for a time T , at the end of which mass (rocket + remaining fuel) is m_1 .

Let $m = m(t)$ be the mass at time t . (rocket + unburned fuel)

Rate of ejection of mass at time t is $-\dot{m}$.

At $t=0$, system = rocket + fuel.

After a time T , system is shown below



Initial linear momentum = $m_0 v_0$.

Final linear momentum of rocket & unburned fuel is $m(\tau)v(\tau)$.

Fuel ejected in time $[t, t+dt]$

$$(-\dot{m}(t))dt$$

with forward velocity at instant of ejection is $v(t) - u$

$$\therefore \text{linear mom. of eject. fuel} = -\dot{m}(v-u)dt.$$

\therefore Total linear mom. of fuel ejected in time interval $[0, \tau]$ is,

$$-\int_0^\tau \dot{m}(v-u) dt.$$

\therefore Conser. of momentum

$$\Rightarrow m_0 v_0 = m(\tau)v(\tau) - \int_0^\tau \dot{m}(v-u) dt.$$

$$\Rightarrow \int_0^\tau \left[\frac{d}{dt}(mv) - \dot{m}(v-u) \right] dt = 0.$$

must hold for any τ during burn,

$$\Rightarrow \frac{d}{dt}(mv) - \dot{m}(v-u) = 0, \quad 0 \leq t \leq \tau.$$

$$\Rightarrow m \frac{dv}{dt} = (-\dot{m})u.$$

$$\Rightarrow \int dv = \int \frac{(-\dot{m})u}{m} dt$$

$$= -u \int \frac{dm}{m} = -u \ln m + C_1,$$

At $t=0$, $v = v_0$, $m = m_0$.

$$\Rightarrow v(t) = v_0 + u \ln \left(\frac{m_0}{m(t)} \right).$$

⊗ Let fuel burns at a constant rate

$$\frac{dM}{dt} = b \text{ \& it lasts for time } T.$$

If mass of vehicle is M_v & mass of fuel is M_f at $t=0$.

$$\Rightarrow M_0 = M_v + M_f.$$

$$M(t) = M_v + M_f \left(1 - \frac{t}{T} \right).$$

$$= M_0 - M_f \frac{t}{T}, \quad 0 \leq t \leq T.$$

$$\& M(t) = M_v \quad \text{for } t \geq T.$$

$$\Rightarrow v = \frac{dx}{dt} = v_0 - u \ln \left(1 - \frac{M_f t}{m_0 T} \right).$$

$$\therefore x = x_0 + v_0 t - u \int_0^t \ln \left(1 - \frac{M_f t}{m_0 T} \right) dt$$

$$\Rightarrow x = x_0 + v_0 t - u \left[\left(t - \frac{m_0 T}{M_f} \right) \ln \left(1 - \frac{M_f t}{m_0 T} \right) - t \right]$$

$$v_{\max} = v(t)$$

$$= v_0 + u \ln \left(\frac{M_0}{M_u} \right).$$

$$= \cancel{v_0 + u \ln}$$

$$= v_0 + u \ln \left(1 + \frac{M_f}{m_0} \right).$$

Under gravity

$$M \ddot{v} = -u \frac{dM}{dt} - Mg.$$

$$\Rightarrow \ddot{v} = -\frac{u}{M} \frac{dM}{dt} - g.$$

$$\Rightarrow v = -u \ln \left(\frac{m(t)}{m_0} \right) - gt.$$

Assume $x=0$ & $v=v_0=0$ at $t=0$.

$$x = vt - \frac{1}{2}gt^2 - \left(t - \frac{m_0 T}{M_f} \right) \ln \left(1 - \frac{M_f t}{m_0 T} \right).$$