Additional conserved quantity

Problem statement

$$V(\vec{r}) = -\frac{k}{r}, \qquad k>0.$$

Show that Laplace-Runge-Lenz vector given by,

$$\vec{W} = \frac{1}{mk}(\vec{L}\times\vec{p}) + \frac{\vec{r}}{r} \quad \text{is a}$$

conserved quantity

$$\dot{\vec{W}} = \frac{1}{mk}(\dot{\vec{L}}\times\vec{p}) + \frac{1}{mk}(\vec{L}\times\dot{\vec{p}}) + \frac{\dot{\vec{r}}}{r} - \frac{\dot{r}\,\vec{r}}{r^2}.$$

$$\dot{\vec{L}} = 0$$

$$\vec{L} = m(\vec{r}\times\dot{\vec{r}})$$

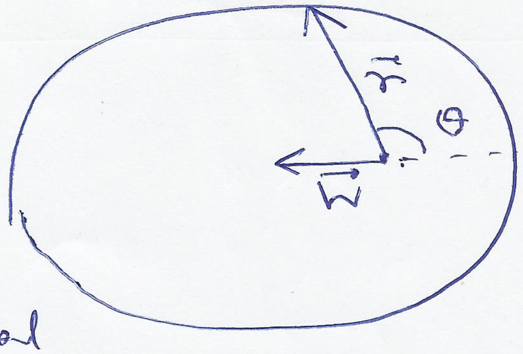
$$\dot{\vec{p}} = m\dot{\vec{v}} = -k\frac{\vec{r}}{r^3}$$

$$\Rightarrow \dot{\vec{W}} = \frac{1}{k}(\vec{r}\times\dot{\vec{r}})\times\left(-\frac{k}{r^3}\vec{r}\right) + \frac{\dot{\vec{r}}}{r} - \frac{\dot{r}}{r^2}\vec{r}.$$

$$= \frac{1}{r^3}\left[-(\vec{r}\times\dot{\vec{r}})\times\vec{r} + r^2\dot{\vec{r}} - (\dot{\vec{r}}\cdot\vec{r})\vec{r}\right]$$

$$= \frac{1}{r^3}\left[(\vec{r}\cdot\dot{\vec{r}})\vec{r} - (\vec{r}\cdot\vec{r})\dot{\vec{r}} + r^2\dot{\vec{r}} - (\dot{\vec{r}}\cdot\vec{r})\vec{r}\right].$$

$= 0. \Rightarrow \vec{W}$ is constant.



Both $\vec{L} \times \vec{p}$ and $\vec{r}$ lies within the orbital plane.

We next obtain the angle ψ between $\vec{W}$ & $\vec{r}$.

$$\vec{W} \cdot \vec{r} = |\vec{W}||\vec{r}| \cos\psi.$$

$$= \frac{1}{mk}(\vec{L} \times \vec{p}) \cdot \vec{r} + \frac{\vec{r} \cdot \vec{r}}{r}.$$

$$= -\frac{1}{mk} \vec{L} \cdot (\vec{r} \times \vec{p}) + r.$$

$$= -\frac{L^2}{mk} + r.$$

$$\Rightarrow r = \frac{L^2/mk}{1 - W\cos\psi}.$$

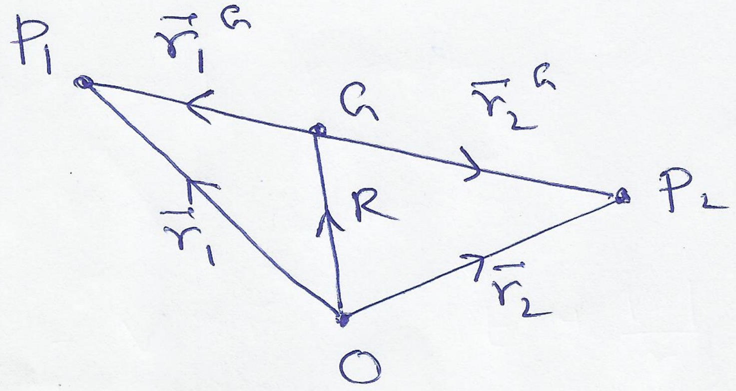
$$\text{or, } \frac{1}{r} = \frac{mk}{L^2} - \frac{W\cos\psi}{L^2/mk}.$$

$$= \frac{1}{(L^2/mk)}(1 - W\cos\psi).$$

$$\therefore W \leftrightarrow e \ \& \ \psi \leftrightarrow \theta + \pi.$$

TWO BODY PROBLEM

Let P_1 & P_2 are moving under their mutual interaction.



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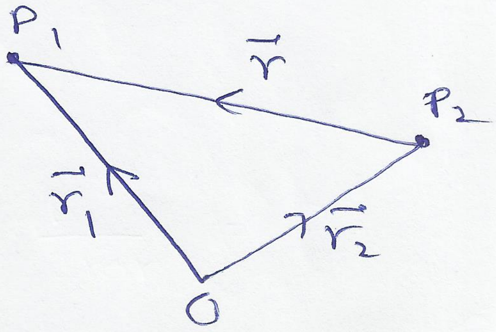


LAB

$$\vec{F}_1 = F(\vec{r})\,\hat{r}.$$

$$\vec{F}_2 = -F(\vec{r})\,\hat{r}.$$

where, $\vec{r} = \vec{r}_1 - \vec{r}_2$

& $\hat{r} = \dfrac{\vec{r}}{|\vec{r}|}$,

$$\Rightarrow m_1 \ddot{\vec{r}}_1 = F(\vec{r})\,\hat{r}.$$

$$m_2 \ddot{\vec{r}}_2 = -F(r)\,\hat{r}.$$

$\boxed{\mu = \left(\dfrac{m_1 m_2}{m_1+m_2}\right) \text{ is reduced mass.}}$

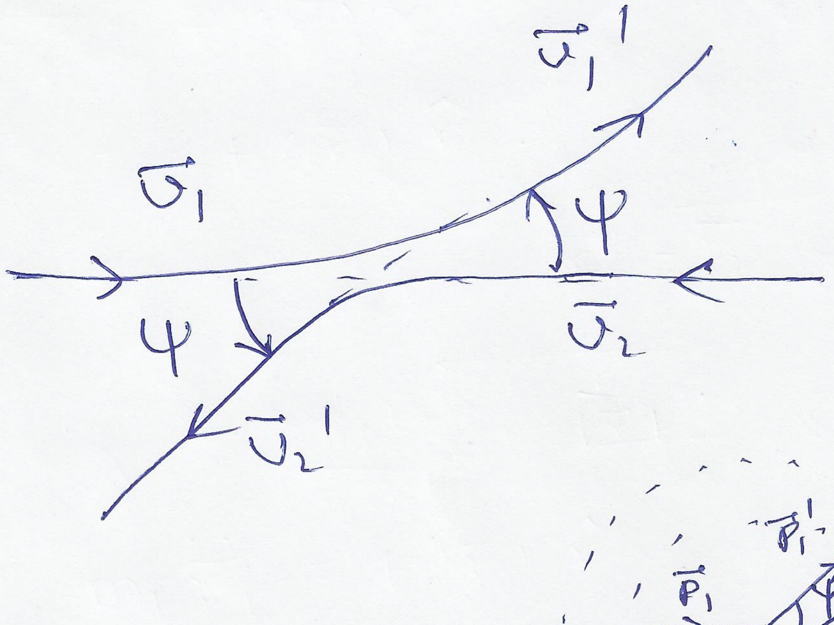
$$\Rightarrow \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = F(r)\,\hat{r}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

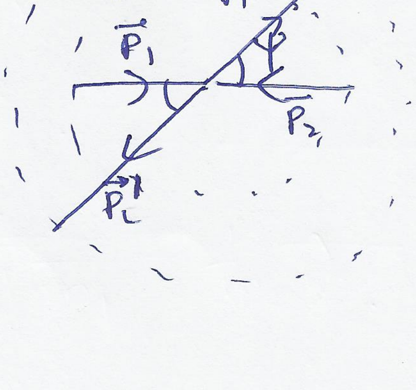
$$\Rightarrow \left(\frac{m_1 m_2}{m_1+m_2}\right) \ddot{\vec{r}} = F(r)\,\hat{r}.$$

& $\bar{r}_1^{\,G} = \left(\dfrac{m_2}{m_1+m_2}\right)\bar{r}$.

$$\bar{r}_2^{\,G} = -\left(\dfrac{m_1}{m_1+m_2}\right)\bar{r}.$$

Notion of zero momentum frame / center of mass frame.





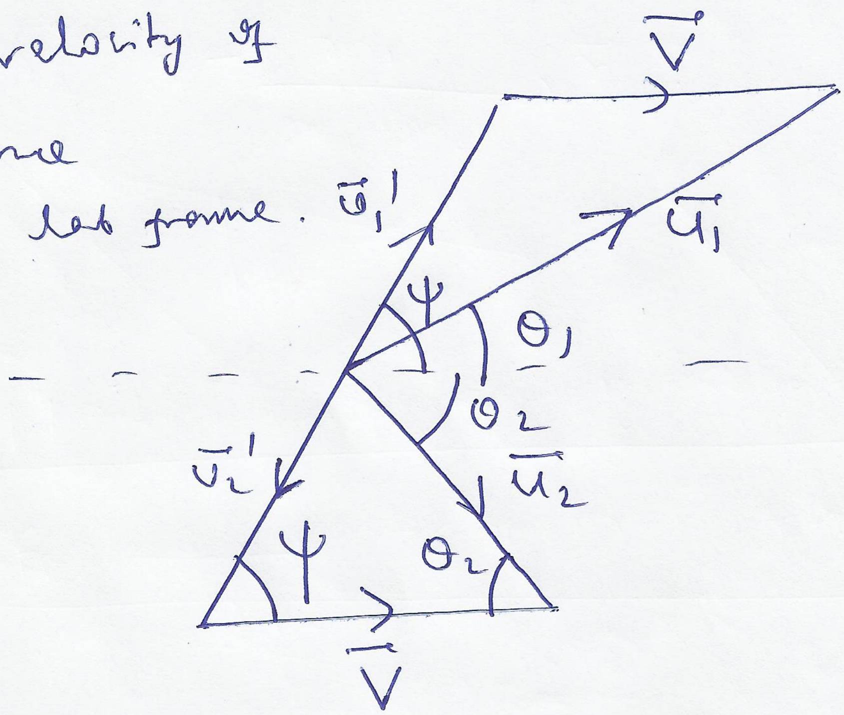
$\bar{p}_1 + \bar{p}_2 = 0.$

$\bar{p}_1' + \bar{p}_2' = 0.$

Elastic Collision

Let the final velocities in lab frame be $\vec{u}_1$ & $\vec{u}_2$

$\vec{V}$ is velocity of ZM frame w.r.t. lab frame.



$$\vec{u}_1 = \vec{v}_1' + \vec{V}$$

$$\vec{u}_2 = \vec{v}_2' + \vec{V}.$$

Show

$$\tan\theta_1 = \frac{\sin\psi}{\cos\psi + (m_1/m_2)}.$$