

Coordinate system  $\rightleftharpoons$  frames of reference

Frame of reference includes all the coordinate systems at rest w.r.t. any particular system.

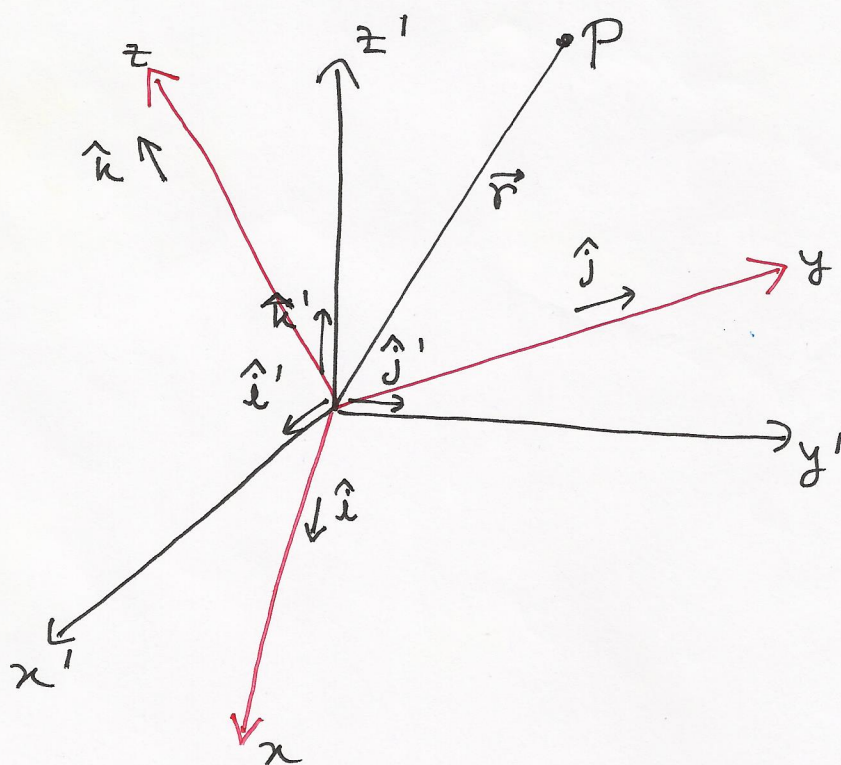
If we use a moving frame of reference, a term occurs in the equation of motion due to the acceleration of the frame.

To have the 'equation of motion' invariant under change of frame, we need to add a term  $-m\ddot{\mathbf{R}}$  to the r.h.s.

This additional term represents a 'fictitious' / 'pseudo' / non-inertial force & has its existence due to the moving frame of reference.

e.g., centrifugal force felt by a person sitting inside a car when the car takes a sudden left turn.

Consider an unprimed coordinate system  $O(x, y, z)$  which is rotating with angular velocity  $\vec{\omega}$  about some instantaneous axis passing through the origin  $O$ . Here,  $O$  is the common origin of the rotating frame of reference  $O(x, y, z)$  & the fixed frame of reference  $O'(x', y', z')$ .



Position vector of a particle  $P$  can be written as,

$$\vec{r} = \hat{i}' x' + \hat{j}' y' + \hat{k}' z'$$

$$\vec{r} = \hat{i} x + \hat{j} y + \hat{k} z$$

Transformation equations from unprimed to primed system can be obtained by taking the dot product of  $\vec{r}$  with  $\hat{i}'$ ,  $\hat{j}'$  and  $\hat{k}'$ .

These are,

$$x' = (\vec{r} \cdot \hat{i}') = (\hat{i} \cdot \hat{i}')x + (\hat{j} \cdot \hat{i}')y + (\hat{k} \cdot \hat{i}')z$$

$$y' = (\vec{r} \cdot \hat{j}') =$$

$$z' = (\vec{r} \cdot \hat{k}') =$$

the dot products are equal to the cosines of angles between the axes.

This is also true in general for any vector  $\vec{V}(t)$ .

$$V = \hat{i} V_x + \hat{j} V_y + \hat{k} V_z = \hat{i}' V_x' + \hat{j}' V_y' + \hat{k}' V_z'$$

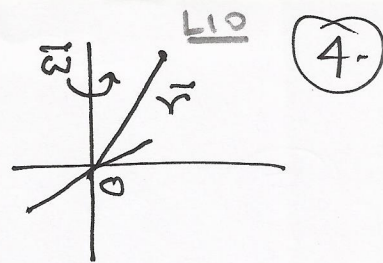
Time derivative would however be different in the two systems.

$$\begin{aligned} \left(\frac{dV}{dt}\right)_{\text{fix}} &= \hat{i}' \dot{V}_x' + \hat{j}' \dot{V}_y' + \hat{k}' \dot{V}_z' \\ &\quad \text{time derivative in rotating system} \\ &= \underbrace{\hat{i} \dot{V}_x + \hat{j} \dot{V}_y + \hat{k} \dot{V}_z}_{\left(\frac{dV}{dt}\right)_{\text{rot}}} + \frac{d\hat{i}}{dt} V_x \\ &\quad + \frac{d\hat{j}}{dt} V_y + \frac{d\hat{k}}{dt} V_z. \end{aligned}$$

(I)

In particular,

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}.$$



(4)

∴ For unit vectors

$\hat{i}, \hat{j}, \hat{k}$  in rotating

frame,

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}, \quad \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}, \quad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}.$$

(II)

Using (I) & (II),

$$\Rightarrow \left(\frac{d\vec{v}}{dt}\right)_{\text{fix}} = \left(\frac{d\vec{v}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{v}.$$

$$\boxed{\left(\frac{d}{dt}\right)_{\text{fix}} = \left(\frac{d}{dt}\right)_{\text{rot}} + \vec{\omega} \times} \quad \text{(III)}$$

Can be operated on any vector.

e.g.,  $\left(\frac{d\vec{\omega}}{dt}\right)_{\text{fix}} = \left(\frac{d\vec{\omega}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{\omega}.$

$$= \left(\frac{d\vec{\omega}}{dt}\right)_{\text{rot}}$$

$$= \dot{\vec{\omega}}.$$

Same in fixed & rotating frames.

(IV)

What about second derivatives? <sup>40</sup> (5)

$$\left. \begin{aligned} \left(\frac{d}{dt}\right)_{\text{fix}} &= \frac{d'}{dt} \\ \left(\frac{d}{dt}\right)_{\text{rot}} &= \frac{d}{dt} \end{aligned} \right\} \text{Notation}$$

$$\Rightarrow \frac{d'^2 \vec{v}}{dt^2} = \frac{d'}{dt} \left( \frac{d' \vec{v}}{dt} \right)$$

$$= \frac{d'}{dt} \left[ \frac{d\vec{v}}{dt} + \vec{\omega} \times \vec{v} \right]$$

$$= \left( \frac{d}{dt} + \vec{\omega} \times \right) \left( \frac{d\vec{v}}{dt} + \vec{\omega} \times \vec{v} \right)$$

$$= \frac{d^2 \vec{v}}{dt^2} + \vec{\omega} \times \frac{d\vec{v}}{dt}$$

$$= \frac{d^2 \vec{v}}{dt^2} + \frac{d\vec{\omega}}{dt} \times \vec{v} + \vec{\omega} \times \frac{d\vec{v}}{dt} + \vec{\omega} \times \vec{\omega} \times \vec{v}.$$

— (V)

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Consider the general case when the origin  $O$  of the rotating coordinate system is also moving w.r.t.  $O'$  of fixed frame.

$$\vec{r}' = \vec{R} + \vec{r}$$

$$\therefore \left(\frac{d\vec{r}'}{dt}\right)_{\text{fix}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{fix}} + \left(\frac{d\vec{r}}{dt}\right)_{\text{fix}}$$

————— (VI.)

Using (III.), (IV.) & (V.) in (VI.),

↓  
show.

$$\left(\frac{d\vec{r}'}{dt}\right)_{\text{fix}} = \left(\frac{d\vec{R}}{dt}\right)_{\text{fix}} + \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}} + \vec{\omega} \times \vec{r}$$

&

$$\left(\frac{d^2\vec{r}'}{dt^2}\right)_{\text{fix}} = \left(\frac{d^2\vec{R}}{dt^2}\right)_{\text{fix}} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{\text{rot}}$$

$$+ 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{\text{rot}}$$

$$+ \vec{\omega} \times \vec{\omega} \times \vec{r}$$

$$+ \frac{d\vec{\omega}}{dt} \times \vec{r}$$

Using symbols  $f$  &  $r$  for fixed & rotating frames,

$$\dot{\vec{r}}'_f = \dot{\vec{R}}_f + \dot{\vec{r}}_r + \vec{\omega} \times \vec{r}$$

$$\ddot{\vec{r}}'_f = \ddot{\vec{R}}_f + \ddot{\vec{r}}_r + 2\vec{\omega} \times \dot{\vec{r}}_r + \vec{\omega} \times \vec{\omega} \times \vec{r} + \dot{\vec{\omega}} \times \vec{r}$$

—VII  
A & B.

$\dot{r}'_f$  &  $\ddot{r}'_f$  = velocity & accl<sup>n</sup>. relative to fixed axes.

$\dot{R}_f$  &  $\ddot{R}_f$  = linear velocity & linear accl<sup>n</sup>. of the origin of the rotating axes.

$\dot{r}_r$  &  $\ddot{r}_r$  = velocity & accl<sup>n</sup>. relative to the rotating axes.

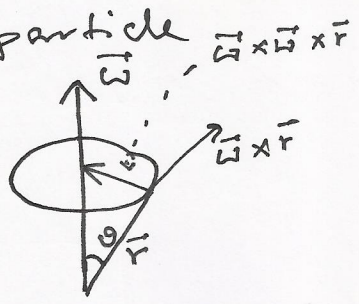
$\vec{\omega}$  = angular velocity of the rotating axes.

$\vec{\omega} \times \vec{r}$  = velocity due to rotation of the axes

$2\vec{\omega} \times \dot{r}_r$  = Coriolis acceleration.

$\vec{\omega} \times \vec{\omega} \times \vec{r}$  = centripetal acceleration

$\dot{\vec{\omega}} \times \vec{r}$  = angular accl<sup>n</sup>. of the particle due to accl<sup>n</sup>. of the rotating axes.



Note: Centripetal (meaning towards the centre)

accl<sup>n</sup>. of the particle situated at point P is directed towards the axes of rotation & is perpendicular to it with mag.

$|\vec{\omega} \times \vec{\omega} \times \vec{r}| = \omega^2 r \sin\theta = \frac{v^2}{r \sin\theta}$

Note:

Coriolis accel<sup>n</sup> is present Lo (8)

only when the particle has a velocity  $\dot{\vec{r}}_r$  in the rotating frame.

Newton's second law,

$$\vec{F} = m \ddot{\vec{r}} \text{ is only}$$

valid in the inertial frame of reference,

$$\vec{F} = \left( \frac{d^2 \vec{r}}{dt^2} \right)_{\text{fix}}.$$

What abt. eq. of motion of a particle in the rotating frame?

Assume: • Angular velocity of rotating system is constant.  $\dot{\vec{\omega}} = 0$ .

• Origins of two frames coincide.  $\vec{R} = 0$ .

Using (VII.) B.,

$$m \left( \frac{d^2 \vec{r}}{dt^2} \right)_{\text{rot.}} = m \left( \frac{d^2 \vec{r}}{dt^2} \right)_{\text{fix}} - 2m \vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{rot.}} - m \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{F}_{\text{eff}}.$$



# L10 (9)

## Forces acting on the particle

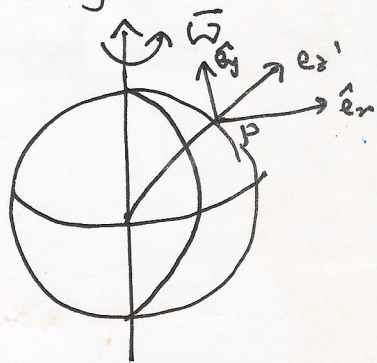
in the rotating frame are:

(i) Real force  $m \left( \frac{d^2 \vec{r}}{dt^2} \right)_{\text{fix}}$

(ii) Centrifugal force  $-m \vec{\omega} \times (\vec{\omega} \times \vec{r})$   
 result of rotation of coordinate axes.

(iii) Coriolis force  $-2m \vec{\omega} \times \left( \frac{d\vec{r}}{dt} \right)_{\text{rot}}$   
 result of motion of the particle in the rotating frame of reference.

Non-inertial forces.



Anti-clockwise  
 right

left  
 clockwise

### Coriolis force

- Climate

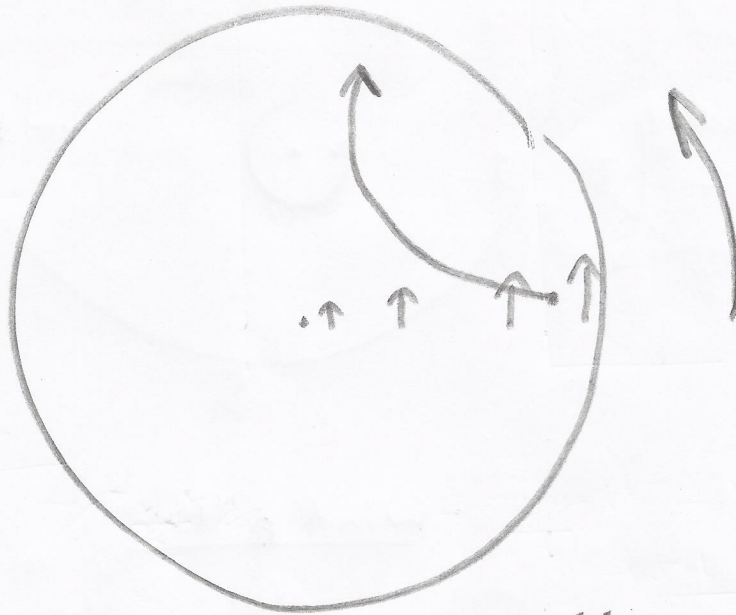
For a velocity

$$1 \text{ km/s} \cong 3600 \text{ km/hr}$$

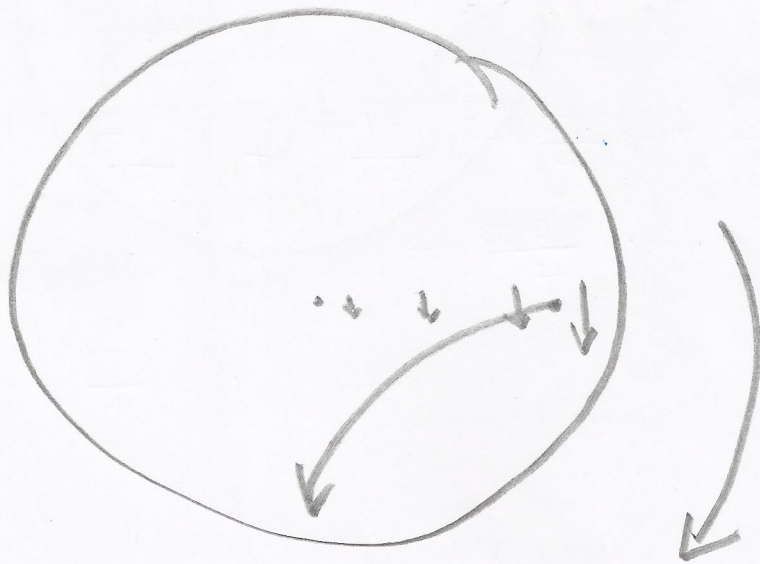
$$\text{Coriolis accel} \cdot \cong 0.15 \text{ m/sec}^2$$

$$\cong 0.015 g$$

- Flight of missiles



Ball on a turntable



Ball on a turntable

Example: ShipTranslation

Moving up & down (heaving)

„ left & right (swaying)

„ forward & backward (surging)

Rotation

Tilt forward & backward (pitching)

Swivels left & right (yawing)

Pivots side to side (rolling)

Robotics

Serial & parallel manipulator system:

Designed to position an end effector

with 6 DOF (3 for translation & 3 for rotation).

Actuator positions



Manipulator configuration

> 6 DOF ;

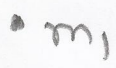

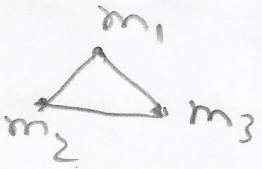
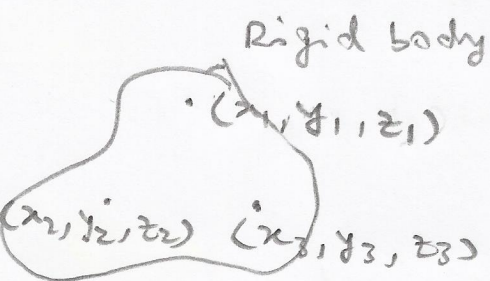
Dean Kamen (2007)

↓  
Robotic arm with 14 DOF.

# DOF

(degrees of freedom)

$$\equiv \# \text{ of coordinates} - \# \text{ of constraint eq.}$$

System	DOF
	3
	$6 - 1 = 5$
	$9 - 3 = 6$
	$9 - 3 = 6$

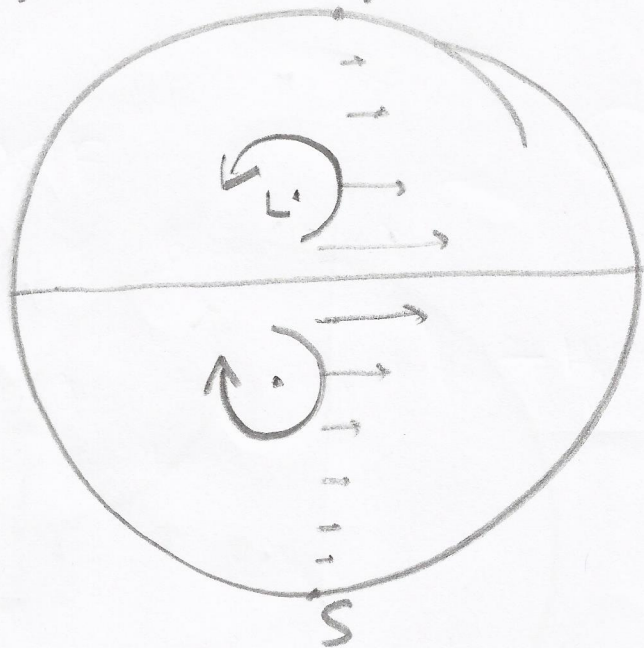
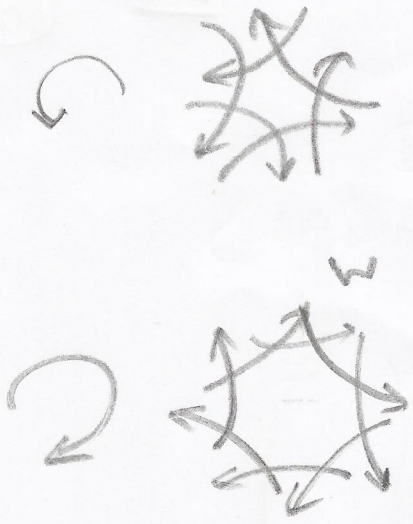
const. eq.

$$\left\{ \begin{aligned} (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 &= d_{ij}^2 \\ \rightarrow 3 \text{ eq.} \end{aligned} \right.$$

Curve to the right

Hurricane

L10 (11)



cyclone  
Typhoon.

E

Tornado

Wind pattern

