EULER'S EQUATION OF MOTION

- **What is Euler’s Equation about?**
- **Euler’s equation derivation- A geometrical approach**

What is Euler’s equation about?

COMPLETE ANALYSIS FOR STABILITY OF A RIGID BODY IN MOTION

SOLUTION OF EULER’S EQUATION GIVES WHETHER MOTION OF RIGID BODY IS BOUND OR UNBOUND

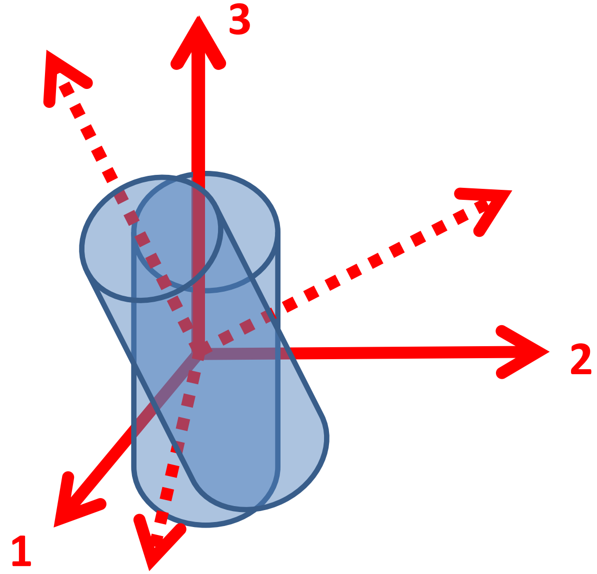
BOUND MOTION = STABLE
UNBOUND MOTION = UNSTABLE

A thought experiment!
To understand and derive Euler equations

A GEOMETRICAL APPROACH…….

'ROTATION OF A CYLINDER'

Another thought experiment!



Initial Conditions…

$$L_1 = I_1\omega_1 \qquad L_2 = I_2\omega_2$$

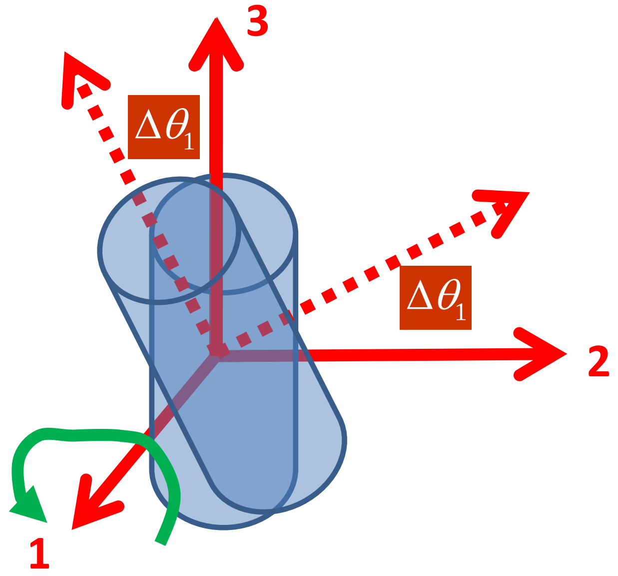
$$L_3 = I_3\omega_3$$

In the time interval Δt, the principal axes rotate away from the 1, 2, 3 axes.

What are the changes in angular momentum in 1,2 and 3 axis?

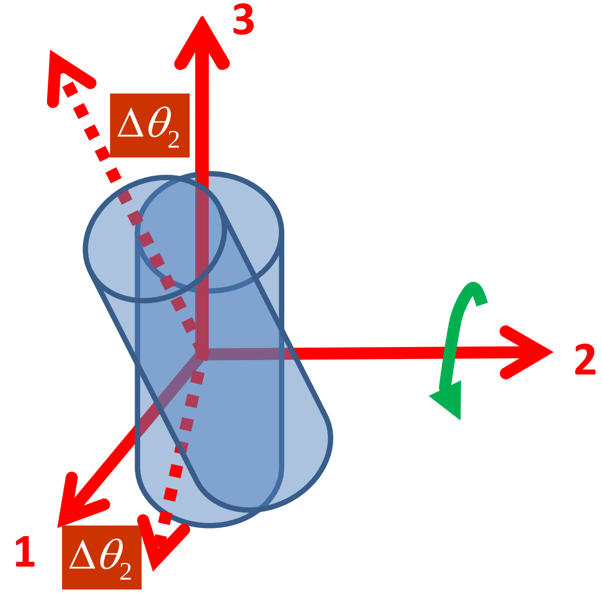
ΔL_1, ΔL_2 and ΔL_3?

ΔL_1 due to rotation along '1' axis

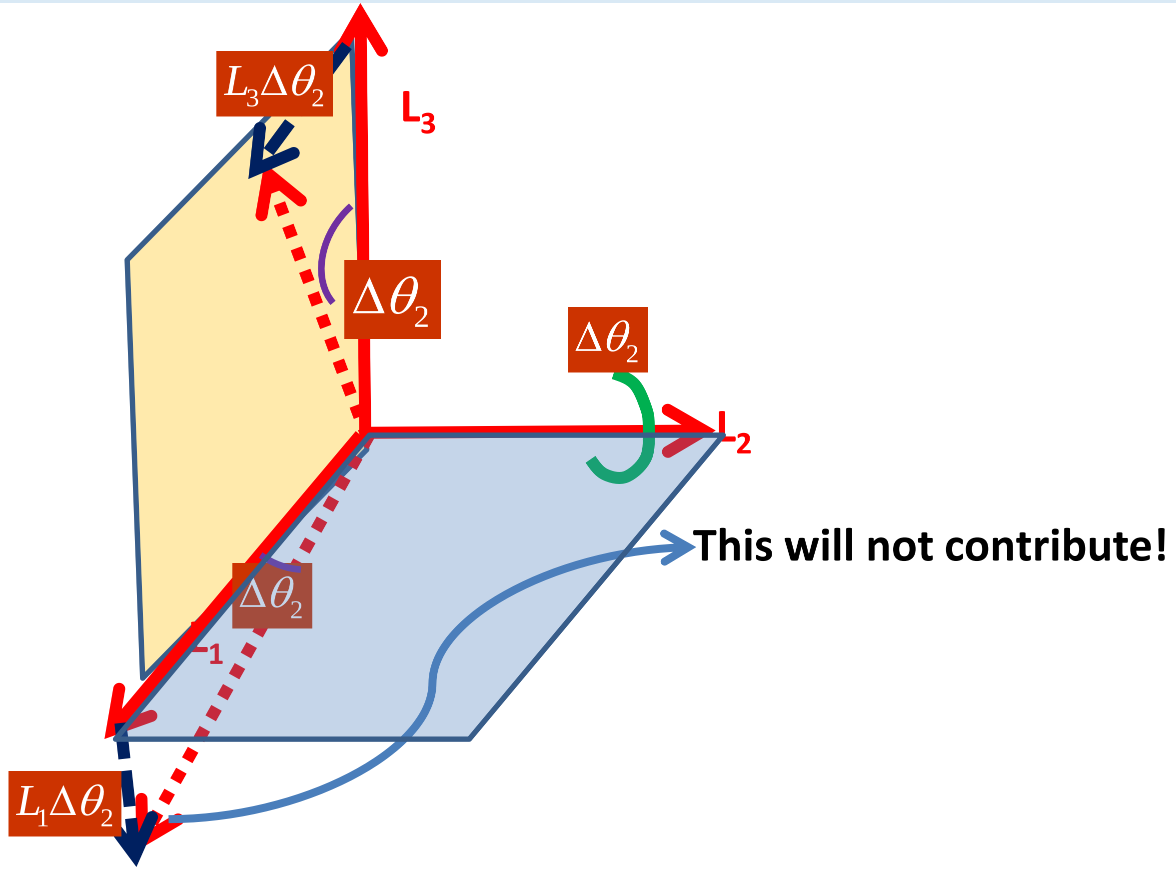


$$\Delta L_1 = I_1 \Delta\omega_1$$

ΔL_1 due to rotation along '2' axis

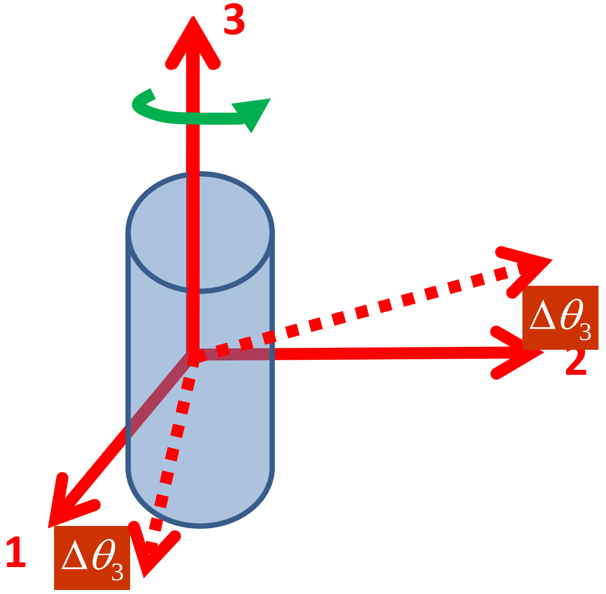


ΔL_1 due to rotation along '2' axis

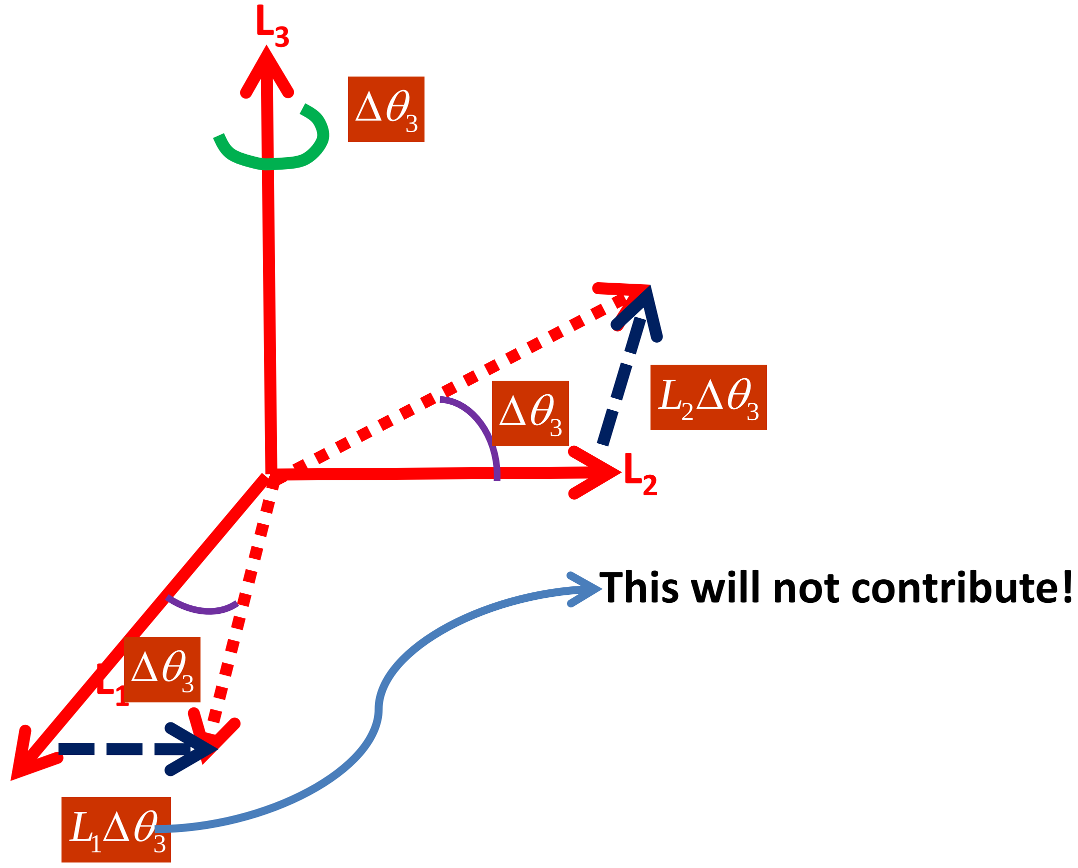


$$\Delta L_1 = I_3\omega_3\Delta\theta_2$$

ΔL_1 due to rotation along '3' axis

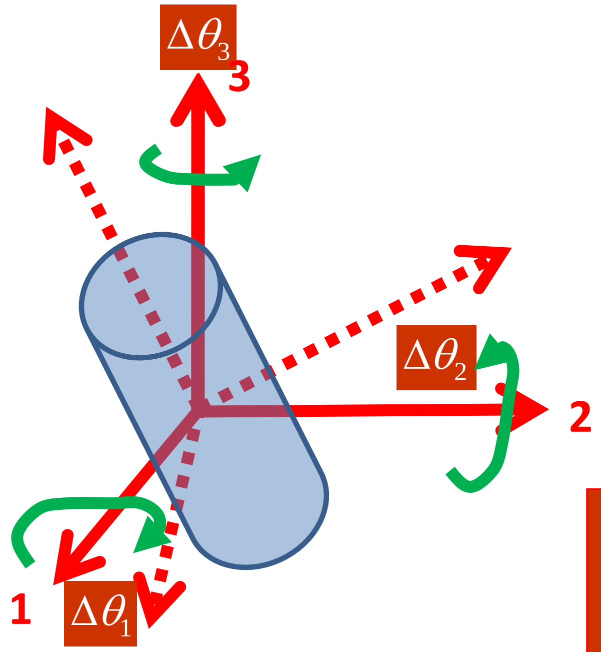


ΔL_1 due to rotation along ‘3’ axis



$$\Delta L_1 = -I_2\omega_2\Delta\theta_3$$

Equation of motion of a Rigid body: Euler's equation



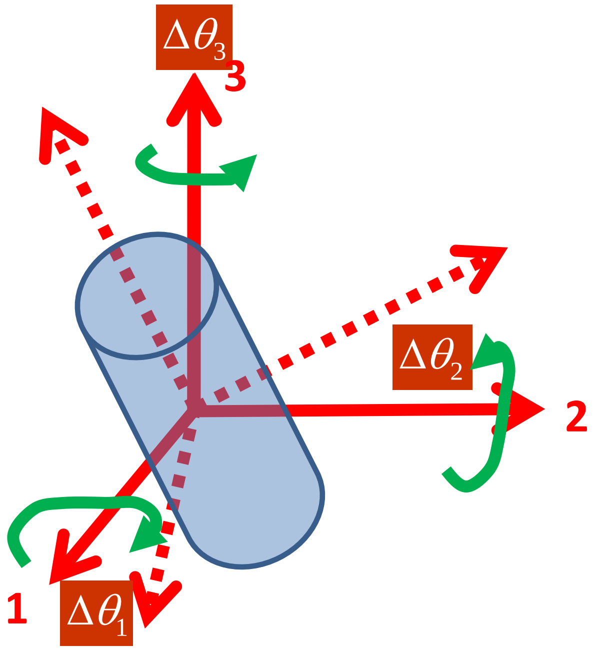
$$\Delta L_1 = I_1 \Delta\omega_1 + I_3 \omega_3 \Delta\theta_2 - I_2 \omega_2 \Delta\theta_3$$

Dividing by Δt **and taking the limit** $\Delta t \to 0$

$$\frac{dL_1}{dt} = I_1 \frac{d\omega_1}{dt} + I_3 \omega_3 \frac{d\theta_2}{dt} - I_2 \omega_2 \frac{d\theta_3}{dt}$$

$$\frac{dL_1}{dt} = I_1 \frac{d\omega_1}{dt} + \left(I_3 - I_2\right) \omega_3 \omega_2$$

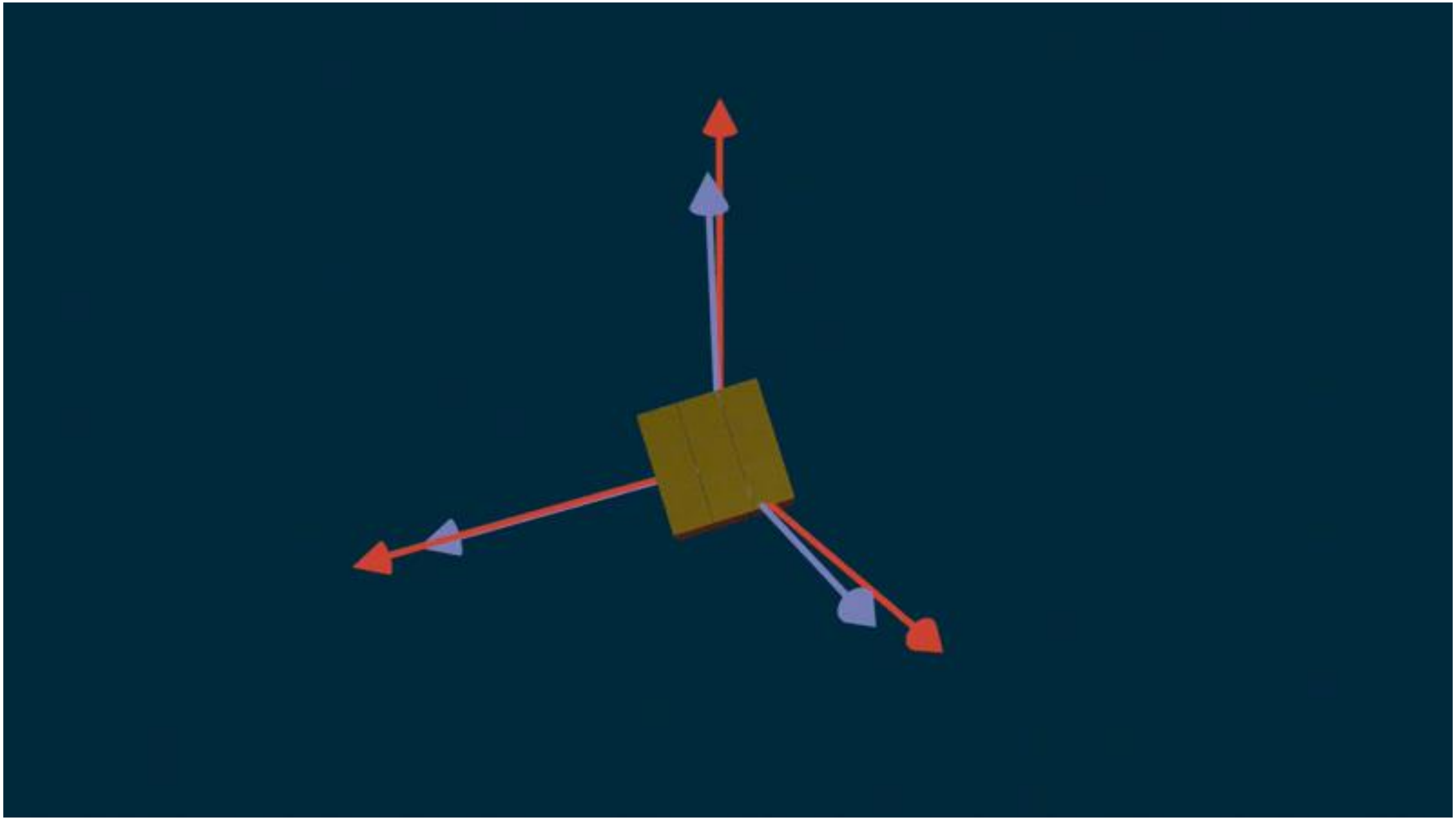
Equation of motion of a Rigid body: Euler's equation



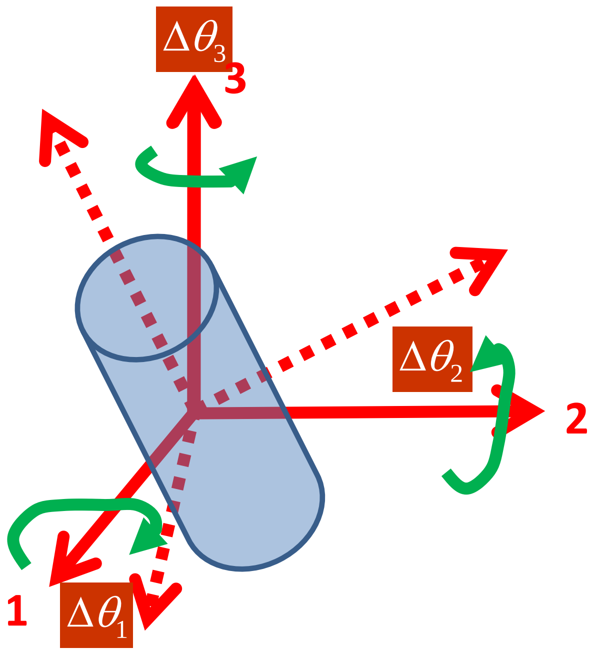
$$\frac{dL_1}{dt} = I_1\frac{d\omega_1}{dt} + \left(I_3 - I_2\right)\omega_3\omega_2$$

$$\frac{dL_2}{dt} = I_2\frac{d\omega_2}{dt} + \left(I_1 - I_3\right)\omega_1\omega_3$$

$$\frac{dL_3}{dt} = I_3\frac{d\omega_3}{dt} + \left(I_2 - I_1\right)\omega_2\omega_1$$



Equation of motion of a Rigid body: Euler's equation



$$\tau_1 = I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_3\omega_2$$

$$\tau_2 = I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_1\omega_3$$

$$\tau_3 = I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_2\omega_1$$

Euler's Equations of rigid body motion

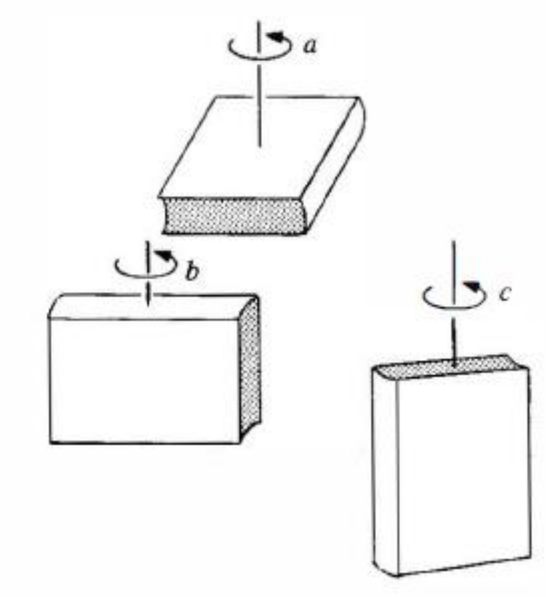
APPLICATION

How Euler’s equations are related to stability?

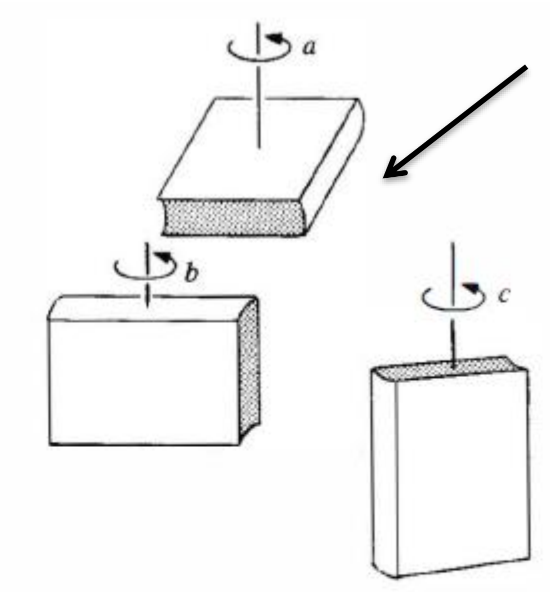
Will learn through some Examples………

Example1

Stability of a book under rotation through its different axis



Book under rotation….



Book is rotated in one of its axis by applying a torque and quickly released. Lets say it has

$$\text{Initially, } \omega_1 = \text{Constant}; \omega_2 = 0; \omega_3 = 0$$

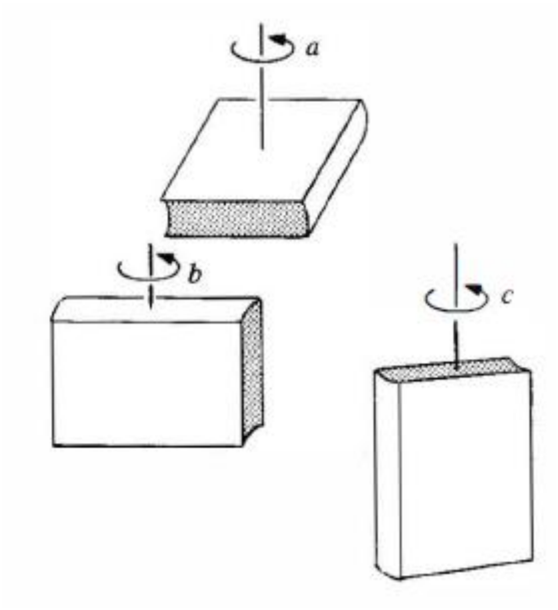
A small perturbation is given …..

$$\omega_1 = \text{Constant}; \omega_2, \omega_3 \multimap \textit{generated}$$

Stability of motion is analyzed by Applying Euler's equation

Guess which of the above configuration will be stable ??

Stability of Rotational motion



$$I_1 \frac{d\omega_1}{dt} + (I_3 - I_2)\,\omega_3\omega_2 = 0 \qquad \textbf{1}$$

$$I_2 \frac{d\omega_2}{dt} + (I_1 - I_3)\,\omega_1\omega_3 = 0 \qquad \textbf{2}$$

$$I_3 \frac{d\omega_3}{dt} + (I_2 - I_1)\,\omega_2\omega_1 = 0 \qquad \textbf{3}$$

$$\text{Initially, } \omega_1 = \text{Constant}; \omega_2 = 0; \omega_3 = 0$$

$$\text{After Perturbation, } I_1 \frac{d\omega_1}{dt} = 0$$

Too small……

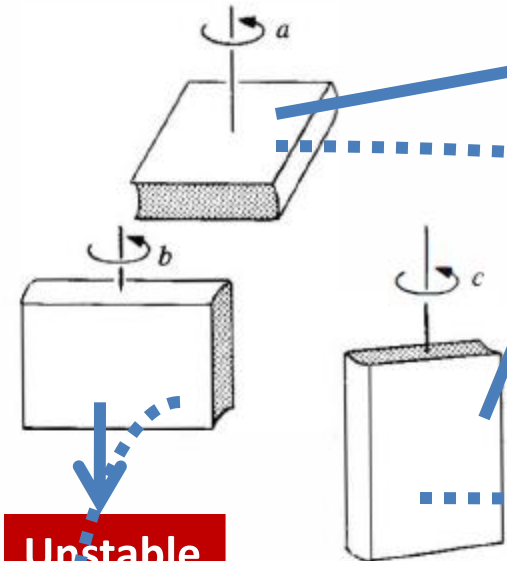
$$\omega_1 = \text{Constant}$$

Since equations 2 and 3 are coupled, we take second derivative of eqn 2 to get equation of motion

$$I_2 \frac{d^2\omega_2}{dt^2} + (I_1 - I_3)\,\omega_1 \frac{d\omega_3}{dt} = 0$$

$$I_2 \frac{d^2\omega_2}{dt^2} - \frac{(I_1 - I_3)(I_2 - I_1)}{I_3}\,\omega_1^2\omega_2 = 0$$

Stability of Rotational motion



Stable

Unstable

$$\frac{d^2\omega_2}{dt^2}+\frac{(I_1-I_3)(I_1-I_2)}{I_2 I_3}\omega_1^2\,\omega_2=0$$

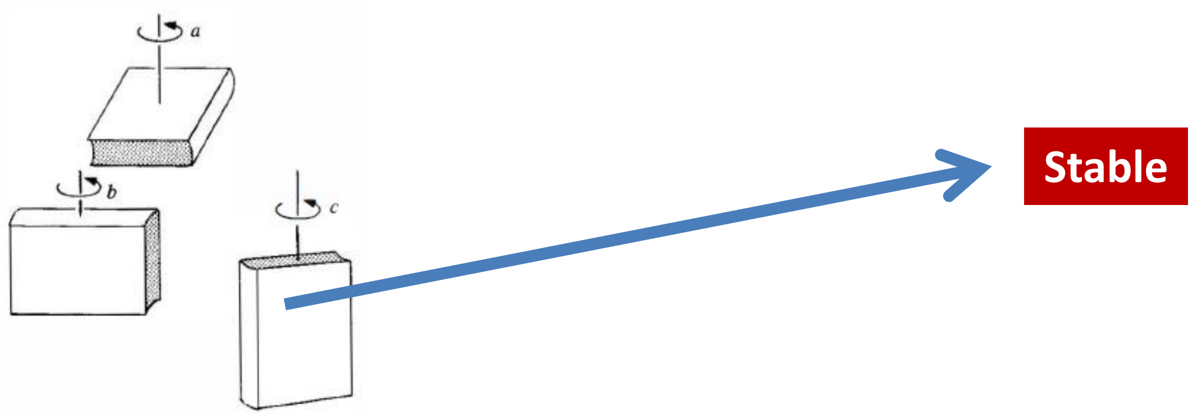
$$\frac{d^2\omega_2}{dt^2}+A\omega_2=0$$

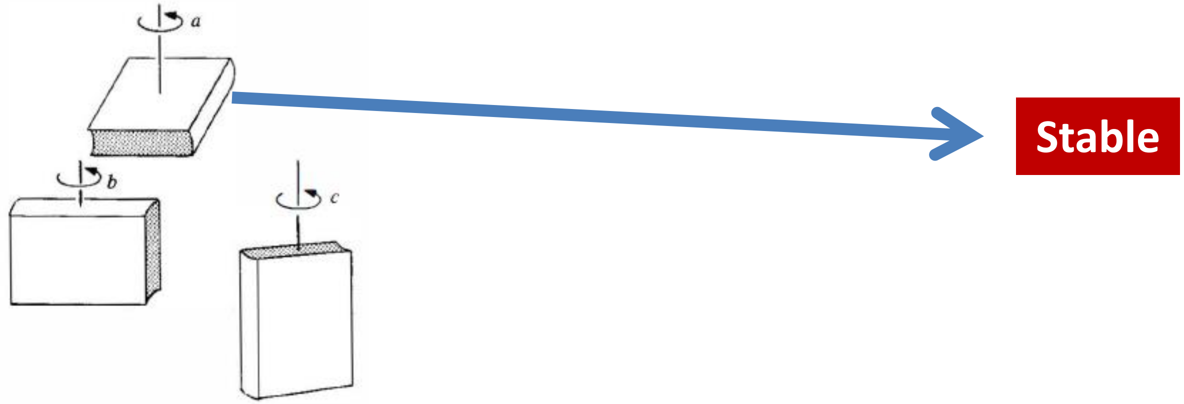
$A>0$ → **Stable** — I_1 is largest or smallest

$A<0$ → **UnStable** — I_1 is intermediate

$A=+\alpha$ ⇒ $\frac{d^2\omega_2}{dt^2}+\alpha\omega_2=0$ ⇒ $\omega_2=C_1\cos\sqrt{\alpha}t+C_2\sin\sqrt{\alpha}t$ — Bound (stable)

$A=-\alpha$ ⇒ $\frac{d^2\omega_2}{dt^2}-\alpha\omega_2=0$ ⇒ $\omega_2=C_2e^{\sqrt{\alpha}t}+C_3e^{-\sqrt{\alpha}t}$ — Unbound (unstable)


a
b
c
Stable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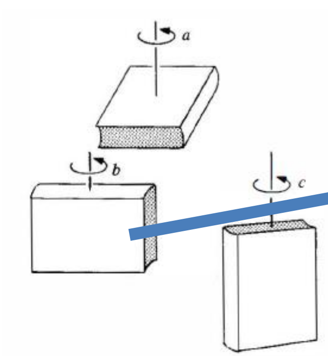

unstable