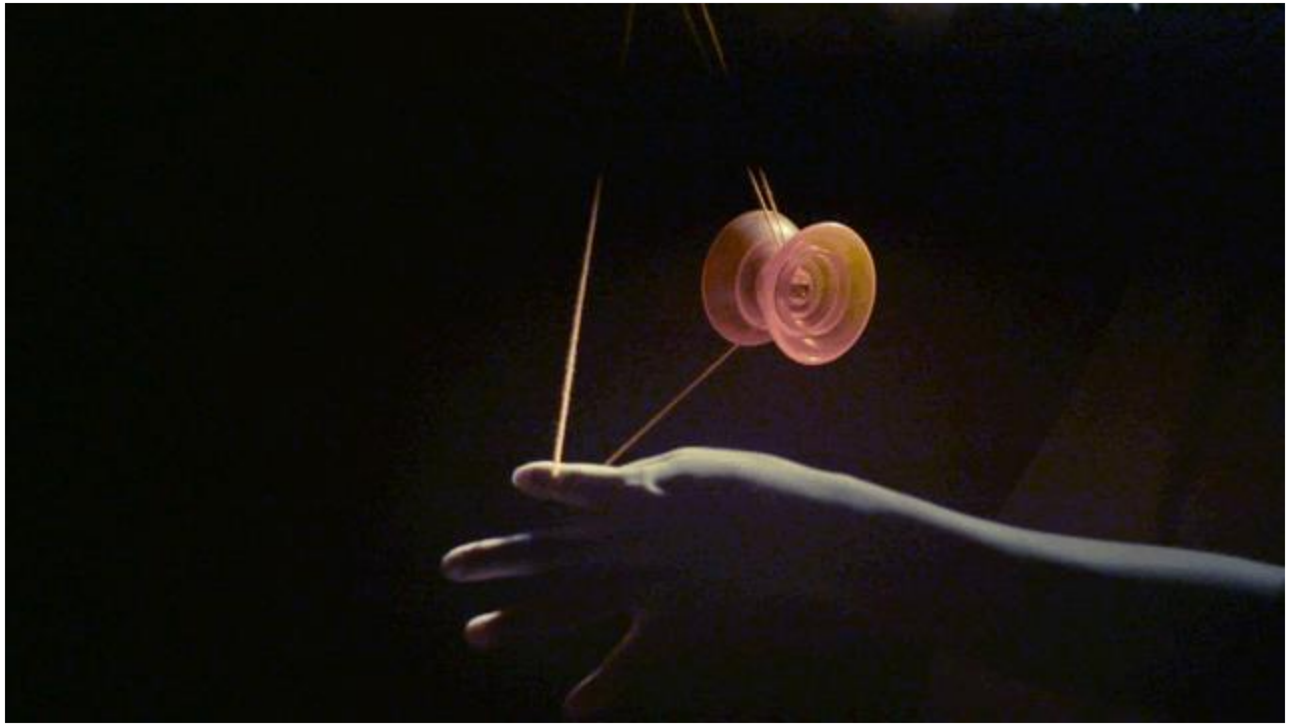
Chapter 4

RIGID BODY IN MOTION



Comparison with Square Laminar Problem

Mathematical approach

Given an Aribitary [I]

Step1: Find the axis of rotation where **L** and **ω** are parallel.

Impose the condition $[I][\vec{\omega}] = \lambda\vec{\omega}$

$$[I][\vec{\omega}] - \lambda[\vec{\omega}] = 0$$

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

Find $\lambda_1, \lambda_2, \lambda_3$

For each λ, corresponding [ω]'s can be found

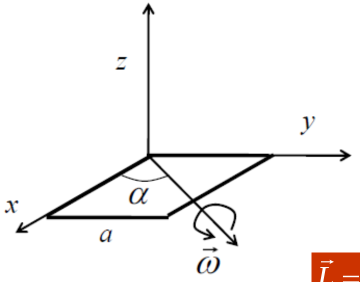
[ω]'s also give the corresponding axis of rotations

Step2: By writing eigen values as diagonal elements chooses Moment of Inertia matrix corresponding to a co-ordinate axis as its rotation axis

MI (Principal axes) $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

Square Laminar Problem

$$[I] = \begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}$$



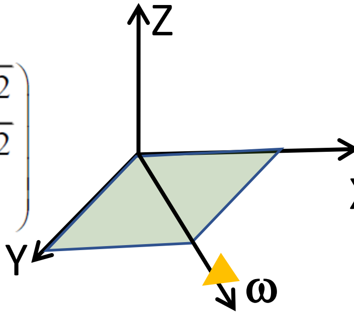
For $\alpha = 45^\circ$,

$$\vec{L} = \left(\frac{1}{12\sqrt{2}}Ma^2\omega, \frac{1}{12\sqrt{2}}Ma^2\omega, 0\right) \text{ and } \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \frac{Ma^2}{12}\begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ \frac{\omega}{\sqrt{2}} \\ 0 \end{pmatrix}$$

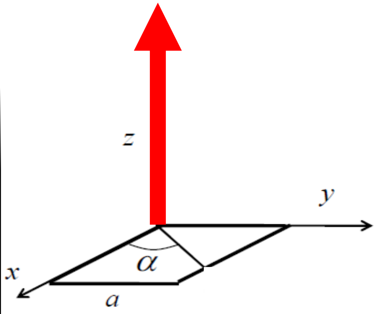
$$\vec{L} = [I][\vec{\omega}] = \lambda\vec{\omega}$$

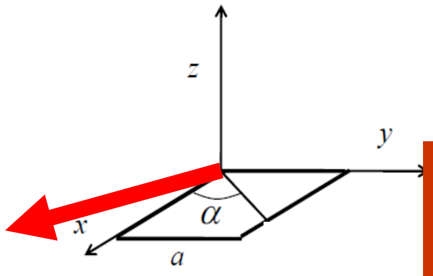
$$\lambda = \frac{Ma^2}{12}$$

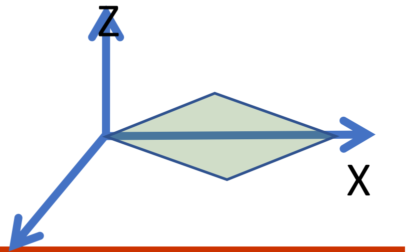


This is one of the λ

Other 2 axis of rotation's for which L and are ω are parallel are







$$\lambda_1 = \frac{2}{3}Ma^2 \qquad \lambda_2 = \frac{7}{12}Ma^2$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

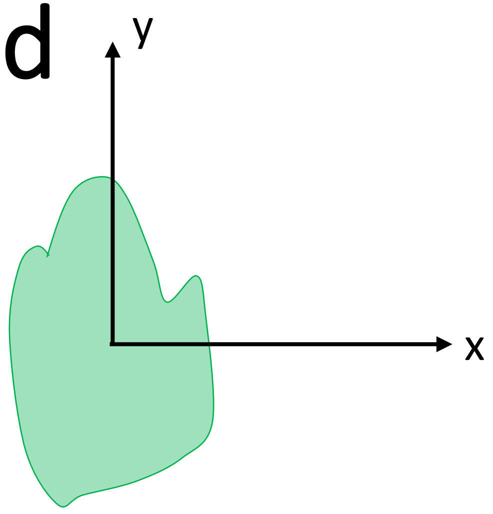
MI (Principal axes)

Will be discussed in Tutorials

Let us try one example

How to find the principal axis of a rigid body?

Given a moment of Inertia matrix $I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ **for some unknown**



This Co-ordinate axis corresponds which the MI matrix given in the problem

Step 1: Find an axis for which **L** is parallel to **ω** for the given [I]

Finding an axis for which **L** is parallel to **ω** for the given [I]

IMPOSING THE CONDITION $[I][\vec{\omega}] = \lambda\vec{\omega}$ **TO THE GIVEN [I]**

$$I = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

BY IMPOSING THIS CONDITION AND FINDING THE UNKNOWNS λ AND ω GIVE INFORMATION ABOUT THE AXIS FOR WHICH L AND ω ARE PARALLEL FOR THE UNKNOWN RIGID BODY

Finding an axis for which **L** is parallel to **ω** for the given [I]

$$[\boldsymbol{I}]\boldsymbol{\omega} - \boldsymbol{\lambda}\boldsymbol{\omega} = \mathbf{0}$$

$$\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \end{bmatrix} = 0$$

Theorem: If **[A][x]=0,** then [A] is non-invertible. This implies A^{-1} does not exist

Hence, |A|=0.

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

Characteristic Equation $\lambda^2 - 4\lambda + 3 = 0$

$$\lambda_1 = 1, \lambda_2 = 3,$$

We can find relations btw ω_x and ω_y

Finding an axis for which **L** is parallel to **ω** for the given [I]

- λ_1, λ_2 are called the Eigen values which satisfies the equation

- [ω]'s are called the Eigen vectors.

- For each Eigen values, Eigen vector can be found.

Eigen vector corresponding to λ $_1$ =1

$$\begin{bmatrix} 2-\lambda_1 & -1 \\ -1 & 2-\lambda_1 \end{bmatrix} \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = 0$$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = 0$$

$$\omega_x - \omega_y = 0$$
$$-\omega_x + \omega_y = 0$$

$$\omega_x = \omega_y = a$$

$$[I][\vec{\omega}] = \lambda_1 \vec{\omega}$$

$$[I][\omega] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} a \\ a \end{bmatrix} = 1 \begin{bmatrix} a \\ a \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to λ $_1$ =1 is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen vector corresponding to λ $_2$ =3

$$\begin{bmatrix} 2-\lambda_2 & -1 \\ -1 & 2-\lambda_2 \end{bmatrix}\begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = 0$$

$$\begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = 0$$

$$\omega_x = -\omega_y$$

$$\text{If } \omega_x = a, \omega_y = -a$$

$$[I][\vec{\omega}] = \lambda_2[\vec{\omega}]$$

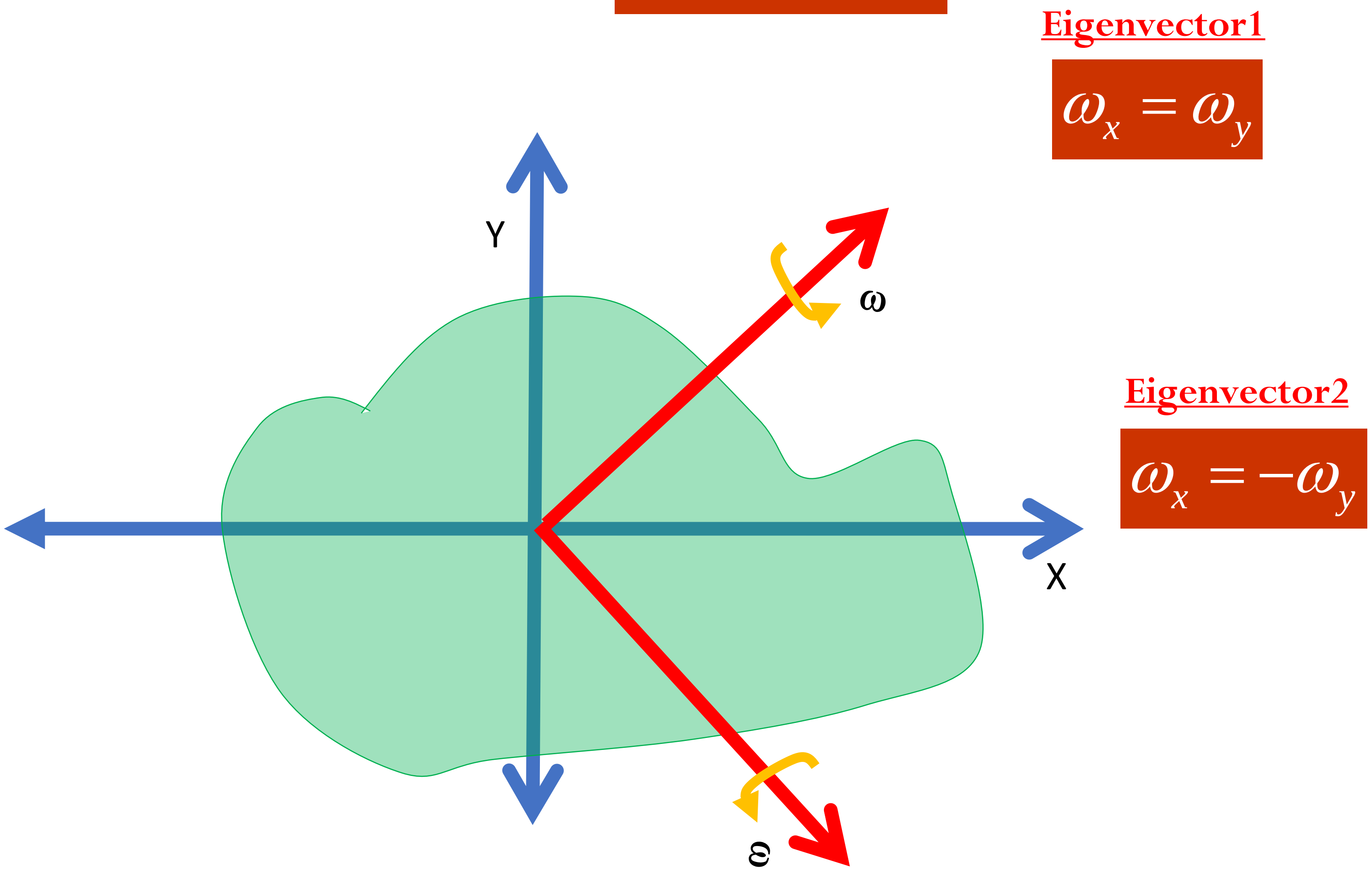
$$[I][\omega] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} a \\ -a \end{bmatrix} = 3\begin{bmatrix} a \\ -a \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

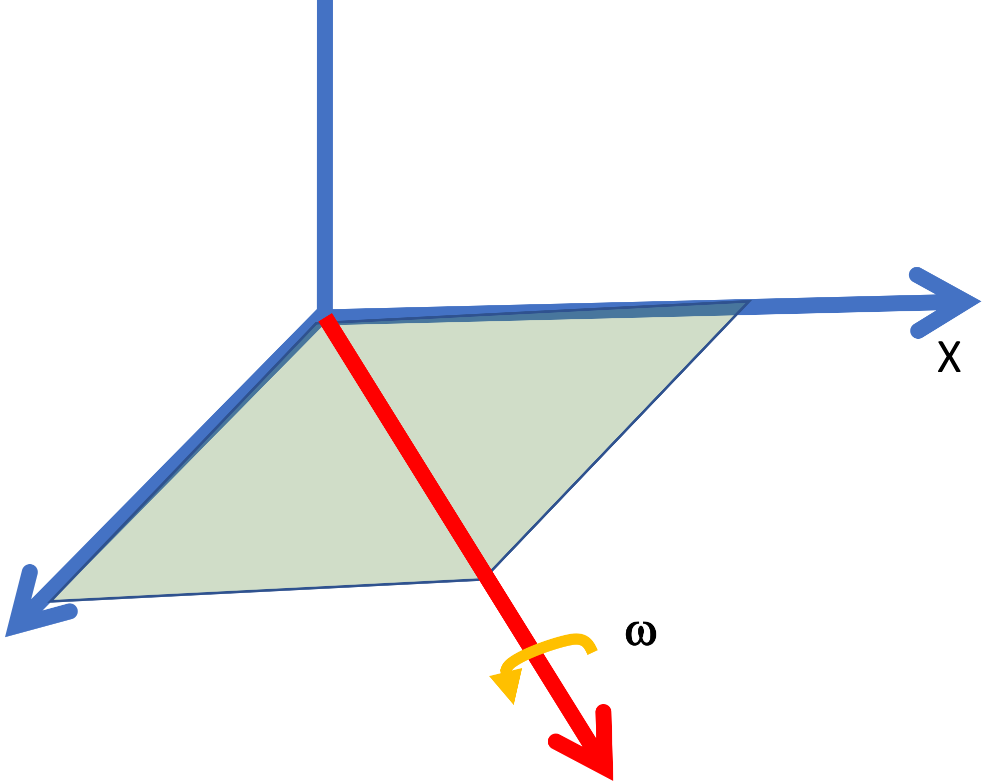
Eigen vector corresponding to λ $_2$ =3 $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Physical meaning of eigenvectors

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$



This Situation is similar to.........**Rotation at 45 degrees**

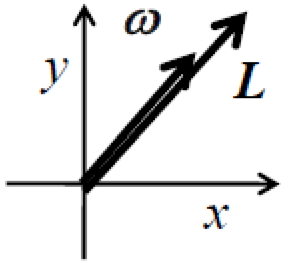


For $\alpha = 45^\circ$,

$$\vec{L} = \left(\frac{1}{12\sqrt{2}} Ma^2 \omega, \frac{1}{12\sqrt{2}} Ma^2 \omega, 0 \right) \quad \text{and} \quad \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\vec{L} = [I][\vec{\omega}] = \lambda \vec{\omega}$$

L and **ω** are parallel!



BUT STILL MI [I] is not diagonal

FINDING THE PRINCIPAL AXIS BY DIAGONALIZATION OF [I]

Step2: Rotating Co-ordinates axis to the axis where L and ω are parallel

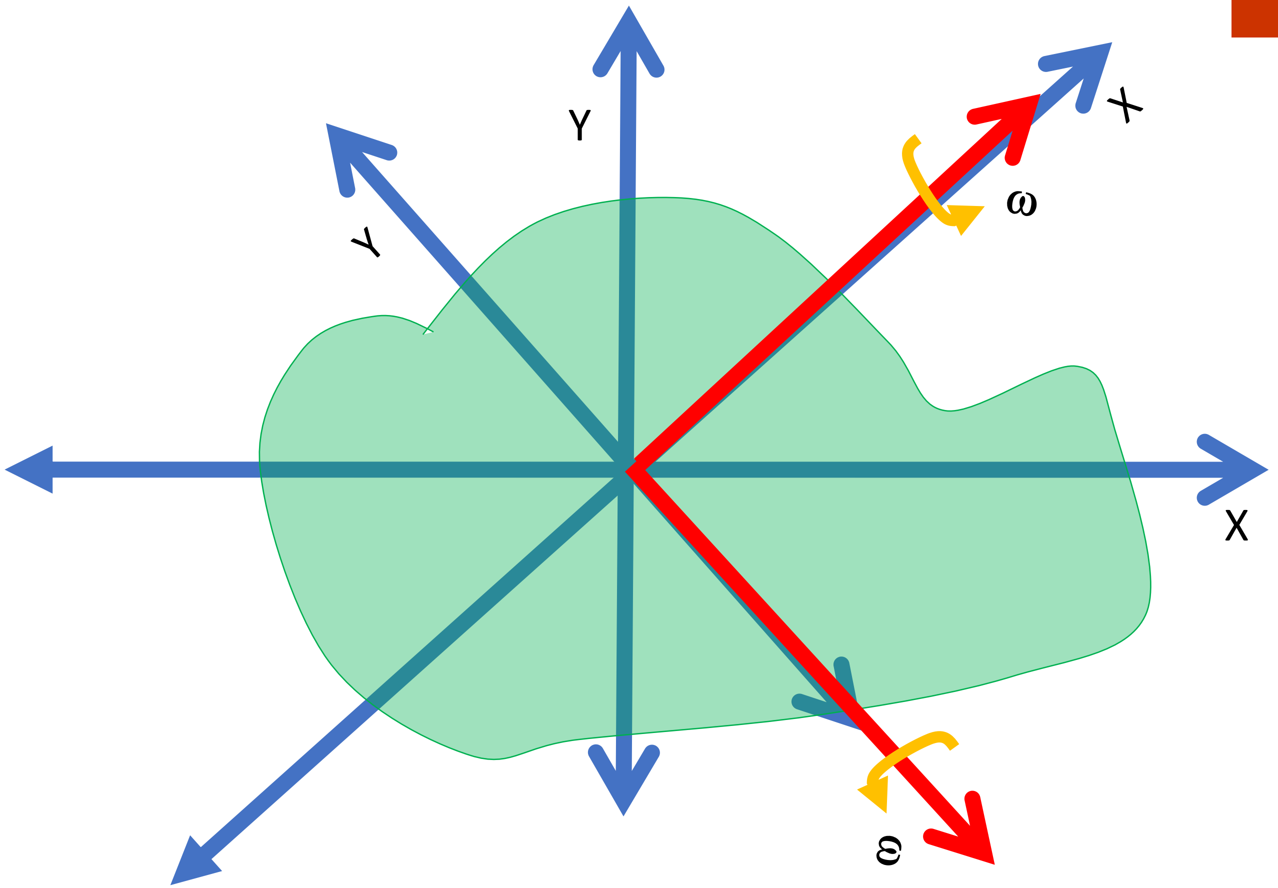
Diagonalization Theorem in mathematics rotates the axis to the axis where L and **ω** are parallel

Diagonalization ensures the rotation axis is along the coordinate axis (Principal axis)

Such Matrix are obtained by writing Eigen Values as diagonal elements

$$I = D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Step2: Rotating Co-ordinates axis to the axis where L and ω are parallel



Eigenvector1

$$\omega_x = \omega_y$$

$$\omega_x = a$$

$$\omega_y = 0$$

Eigenvector2

$$\omega_x = -\omega_y$$

$$\omega_y = a$$

$$\omega_x = 0$$

Moment of Inertia matrix for principal axis

Moment of Inertia Matrix in the principal axes $I = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$$[I][\vec{\omega}] = \lambda_1 \vec{\omega}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a \\ 0 \end{bmatrix} = 1\begin{bmatrix} a \\ 0 \end{bmatrix}$$

Eigen Values in the principal axis are $\lambda_1 = 1, \lambda_2 = 3,$

$$[I][\vec{\omega}] = \lambda_2 \vec{\omega}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ a \end{bmatrix} = 3\begin{bmatrix} 0 \\ a \end{bmatrix}$$

Eigen Vectors in the principal axis are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Angular momentum equations are

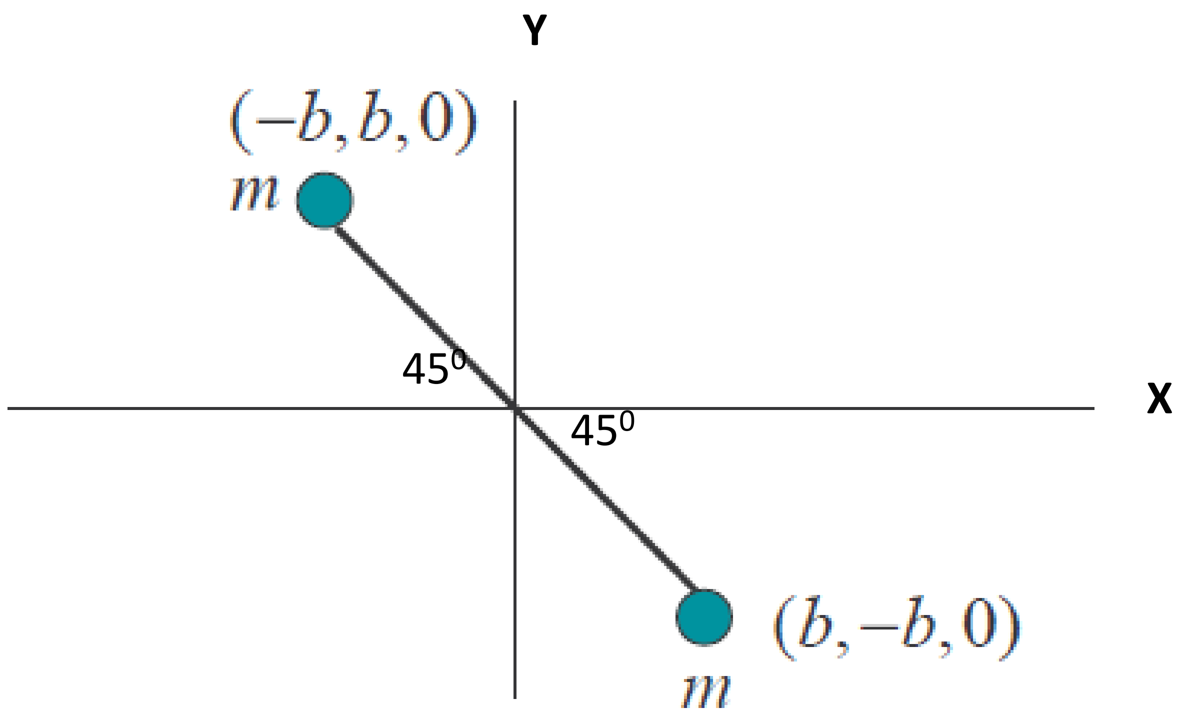
$$L = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Another example to find principal axes

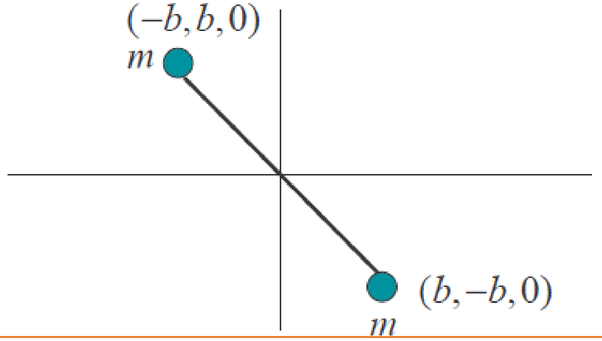
ROTATING DUMBBELL

For the case of a dumb-bell
A. Find the Moment of Inertia matrix
B. Find the principal axis corresponding to it



A. Moment of Inertia matrix

$$[I] = \begin{pmatrix} \int (y^2+z^2)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int (z^2+x^2)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int (x^2+y^2)dm \end{pmatrix}$$



$$\mathbf{I} = m\begin{bmatrix} 2b^2 & 2b^2 & 0 \\ 2b^2 & 2b^2 & 0 \\ 0 & 0 & 4b^2 \end{bmatrix} = 2b^2 m \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Finding the Eigenvalues

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 2mb^2-\lambda & 2mb^2 & 0 \\ 2mb^2 & 2mb^2-\lambda & 0 \\ 0 & 0 & 4mb^2-\lambda \end{vmatrix} = 0$$

Eigenvalues

$$\begin{vmatrix} 2mb^2-\lambda & 2mb^2 & 0 \\ 2mb^2 & 2mb^2-\lambda & 0 \\ 0 & 0 & 4mb^2-\lambda \end{vmatrix} = 0$$

$$(4mb^2-\lambda)\left[\left(2mb^2-\lambda\right)^2-\left(2mb^2\right)^2\right]=0$$

$$(4mb^2-\lambda)\left[\lambda^2-4mb^2\lambda\right]=0$$

$$(4mb^2-\lambda)(\lambda)\left(\lambda-4mb^2\right)=0$$

$$\lambda_1=\lambda_2=4mb^2,\lambda_3=0$$

Moment of Inertia matrix along principal axis

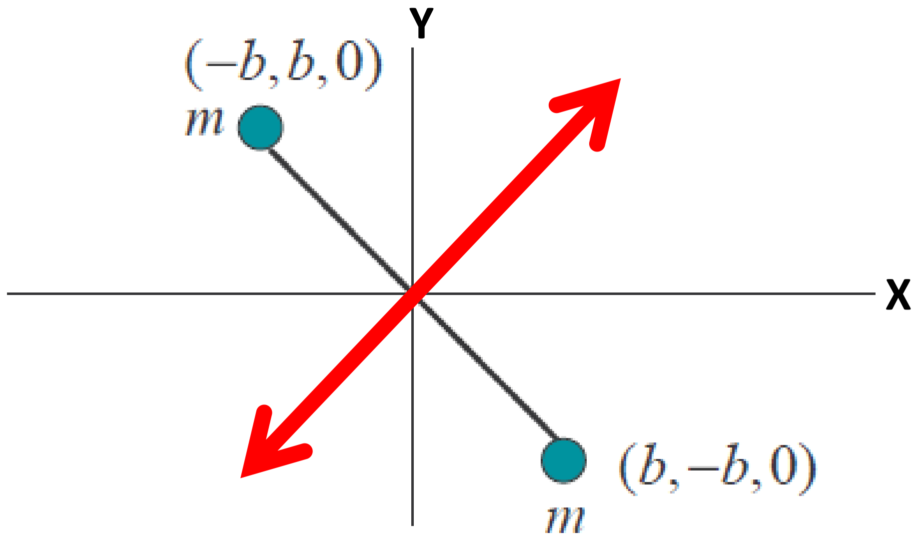
$$\lambda_1 = \lambda_2 = 4mb^2, \lambda_3 = 0$$

$$[I] = \begin{pmatrix} 4mb^2 & 0 & 0 \\ 0 & 4mb^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Eigen vector corresponding to $\lambda_1 = \lambda_2 = 4mb^2$

$$\begin{pmatrix} 2mb^2 - \lambda_1 & 2mb^2 & 0 \\ 2mb^2 & 2mb^2 - \lambda_1 & 0 \\ 0 & 0 & 4mb^2 - \lambda_1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\begin{pmatrix} -2mb^2 & 2mb^2 & 0 \\ 2mb^2 & -2mb^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$



$$-2mb^2\omega_x + 2mb^2\omega_y = 0$$

$$2mb^2\omega_x - 2mb^2\omega_y = 0$$

$$\omega_x = \omega_y$$

$$\omega_z = Anything = \alpha$$

$$\vec{\omega} = \omega_x \hat{e}_x + \omega_x \hat{e}_y + \alpha \hat{e}_z$$

Eigen vector: $\begin{bmatrix} 1 \\ 1 \\ \alpha \end{bmatrix}$

Eigenvectors corresponding $\lambda_3 = 0$

$$\begin{pmatrix} 2mb^2-\lambda_3 & 2mb^2 & 0 \\ 2mb^2 & 2mb^2-\lambda_3 & 0 \\ 0 & 0 & 4mb^2-\lambda_3 \end{pmatrix}\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix}=0 \qquad for\ \lambda_3 = 0$$

$$\omega_x = -\omega_y$$

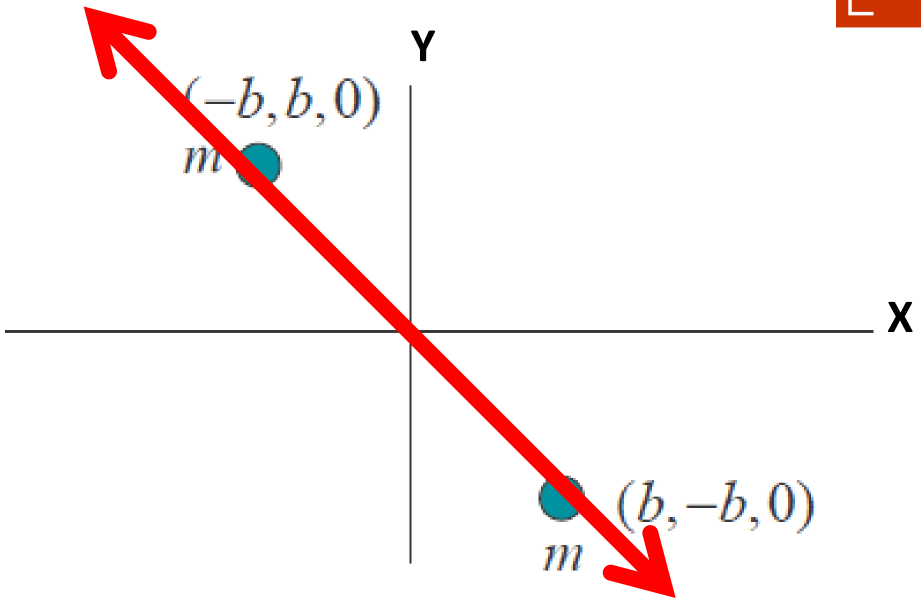
$$\omega_z = 0$$

Eigenvector corresponding to λ_{3}:

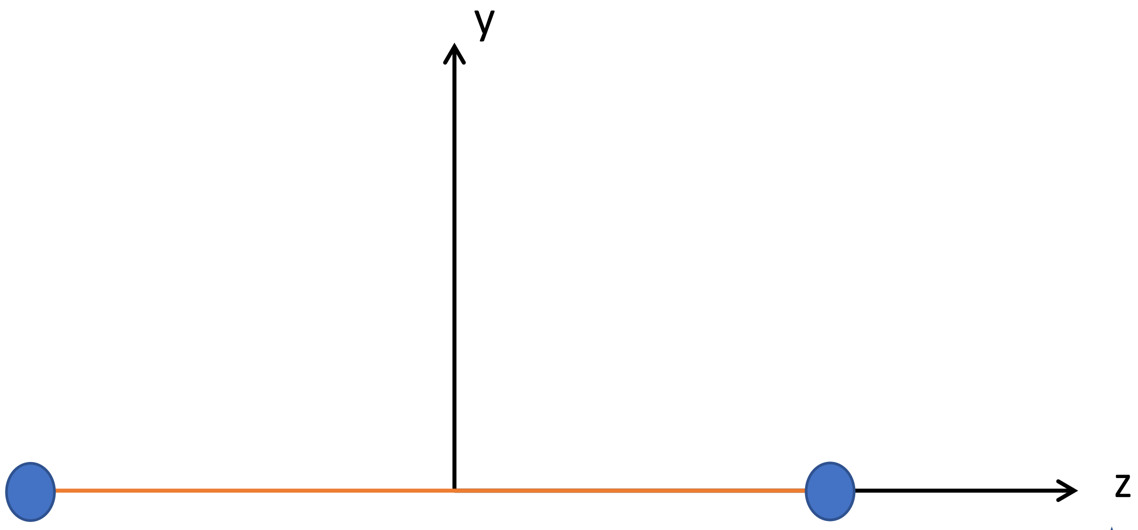
$$\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\vec{L} = [I][\vec{\omega}] = \lambda_3\vec{\omega} = 0$$

$$\vec{\omega} = \omega_x\hat{e}_x - \omega_x\hat{e}_y + 0\hat{e}_z$$



When Moment of Inertia matrix becomes diagonal?



$$I = \begin{bmatrix} 4mb^2 & 0 & 0 \\ 0 & 4mb^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Principal axis