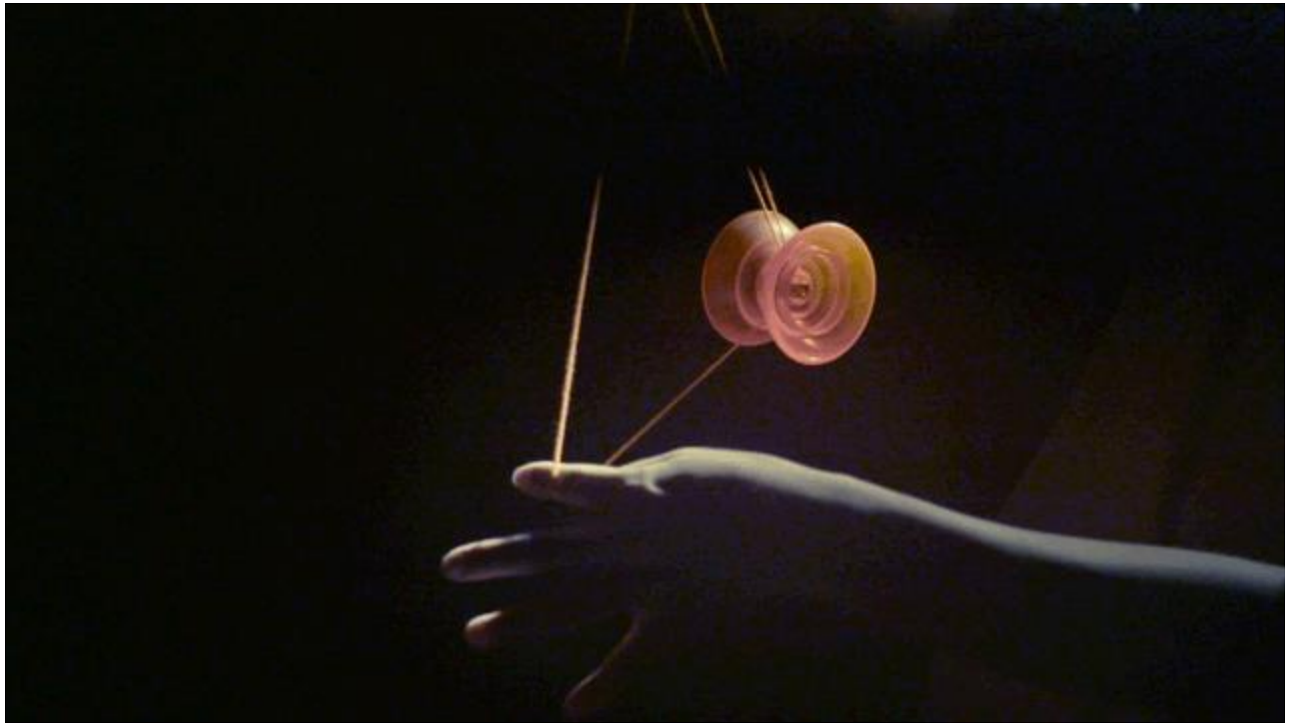
Chapter 4

RIGID BODY IN MOTION



How to find principal axis????

For simple geometries it is easy to find by intuition but not for any general case

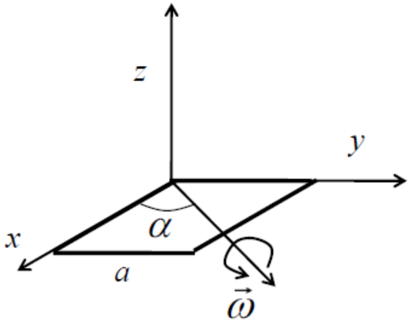
How to find diagonalized MI tensor??

Mathematical approach!

Square Laminar Problem

How to find principal axis????

$$[I]=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$



$$\vec{L}=[I][\vec{\omega}]$$

$$[I][\vec{\omega}]=\lambda\vec{\omega}$$

Unknowns

Step: 1

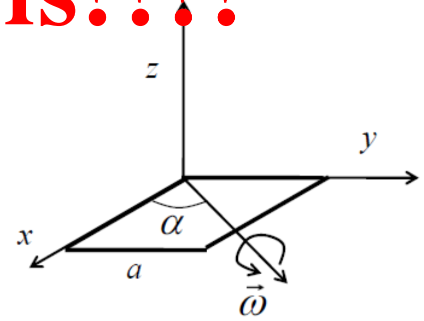
Find the axis of rotation where **L** and **ω** are parallel.

WE IMPOSE THE CONDITION, TO MAKE L AND ω PARALLEL

$$[I][\vec{\omega}]=\lambda\vec{\omega}$$

Find the unknowns ω and λ

How to find principal axis????



$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$\vec{L} = [I][\vec{\omega}]$$

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

Unknowns

$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \lambda \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

Insert a unit matrix

How to find principal axis????

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

Theorem: If **[A][x]=0,** then [A] is non-invertible. This implies A^{-1} does not exist

Hence, |A|=0.

MA102 Mathematics - II

MA102	Mathematics - II	3-1-0-8	Pre-requisites: nil

Linear Algebra: Vector spaces (over the field of real and complex numbers). Systems of linear equations and their solutions. Matrices, determinants, rank and inverse. Linear transformations. Range space and rank, null space and nullity. Eigenvalues and eigenvectors. Similarity transformations. Diagonalization of Hermitian matrices. Bilinear and quadratic forms.

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

Characteristic Equation

How to find principal axis????

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

Solving characteristic equation result in $\lambda_{1,}\ \lambda_2,\ \lambda_{3.}$

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

Substitute $\lambda_{1,}\ \lambda_2,\ \lambda_3$ separately and solve for ωs

Note!

- $\lambda_{1,}$ λ_2, λ_3 are called the Eigen values which satisfies the equation

$$[I][\vec{\omega}] = \lambda\vec{\omega}$$

- [ω]'s are called the Eigen vectors.

- For each Eigen values, Eigen vector can be found.

Step2: How to find diagonalized MI tensor??

Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the principal axes

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

By writing eigen values as diagonal elements chooses Moment of Inertia matrix corresponding to a co-ordinate axis as its rotation axis

Step2: How to find diagonalized MI tensor??

Following Diagonalization Theorem,

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Diagonalization ensures the rotation axis is along the coordinate axis (Principal axis)

Diagonalization Theorem in mathematics rotates the co-ordinate axis to the axis where L and **ω** are parallel

(Will be taught in MA102)

MA102 Mathematics - II

MA102	Mathematics - II	3-1-0-8	Pre-requisites: nil

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Comparison with Square Laminar Problem

Mathematical approach

Given an Aribitary [I]

Step1: Find the axis of rotation where **L** and **ω** are parallel.

Impose the condition $[I][\vec{\omega}] = \lambda\vec{\omega}$

$$[I][\vec{\omega}] - \lambda[\vec{\omega}] = 0$$

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\begin{vmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{vmatrix} = 0$$

Find $\lambda_1, \lambda_2, \lambda_3$

For each λ, corresponding [ω]'s can be found

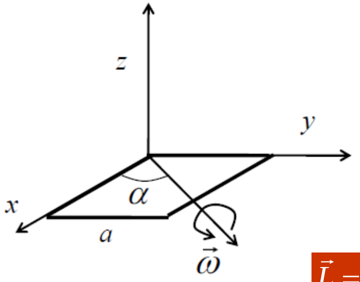
[ω]'s also give the corresponding axis of rotations

Step2: By writing eigen values as diagonal elements chooses Moment of Inertia matrix corresponding to a co-ordinate axis as its rotation axis

MI (Principal axes) $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

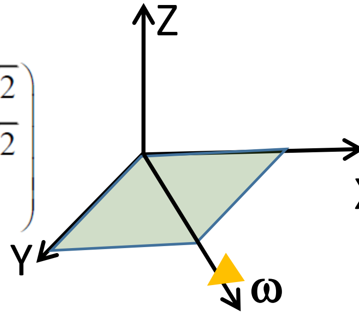
Square Laminar Problem

$$[I] = \begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}$$



For $\alpha = 45^\circ$,

$$\vec{L} = \left(\frac{1}{12\sqrt{2}}Ma^2\omega, \frac{1}{12\sqrt{2}}Ma^2\omega, 0\right) \quad \text{and} \quad \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$



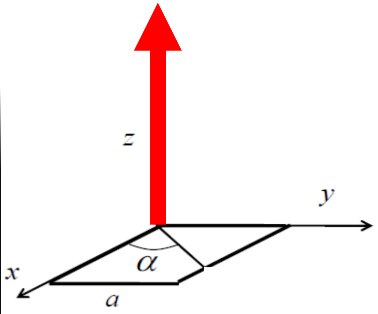
$\vec{L} = [I][\vec{\omega}] = \lambda\vec{\omega}$

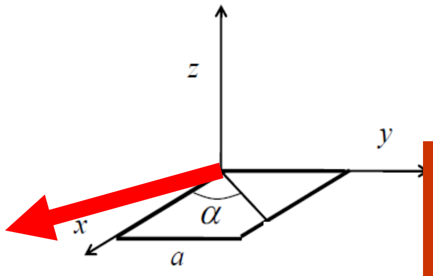
$\lambda = \frac{Ma^2}{12}$

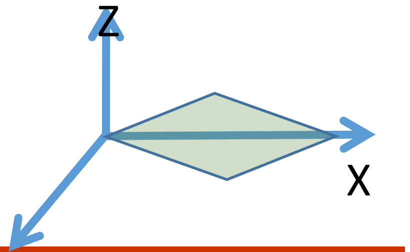
$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \frac{Ma^2}{12}\begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ \frac{\omega}{\sqrt{2}} \\ 0 \end{pmatrix}$$

This is one of the λ

Other 2 axis of rotation's for which L and are ω are parallel are







$\lambda_1 = \frac{2}{3}Ma^2$

$\lambda_2 = \frac{7}{12}Ma^2$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

MI (Principal axes)

Will be discussed in Tutorials