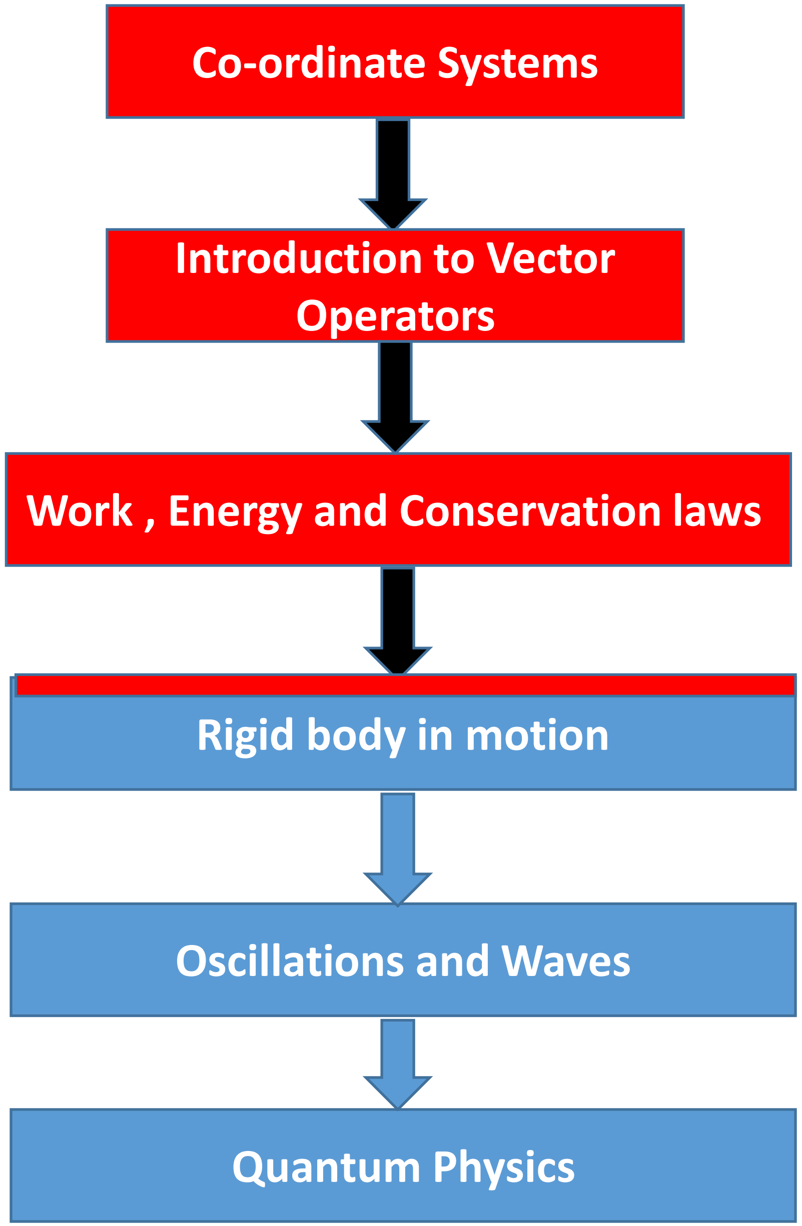
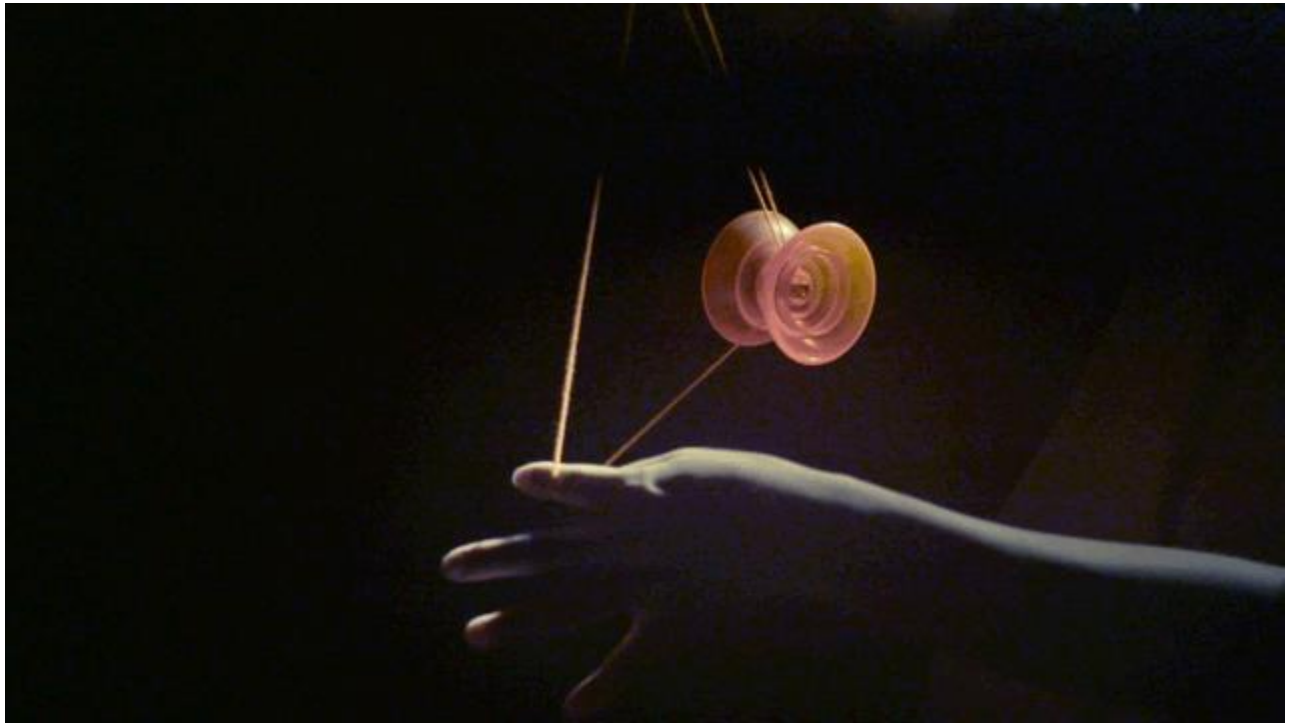
Highlights of the course



Chapter 4

RIGID BODY IN MOTION



Moment of Inertia Matrix

$$[I]=\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{xx}=\sum m_j\left(y_j^2+z_j^2\right)$$

$$I_{xy}=-\sum m_j x_j y_j$$

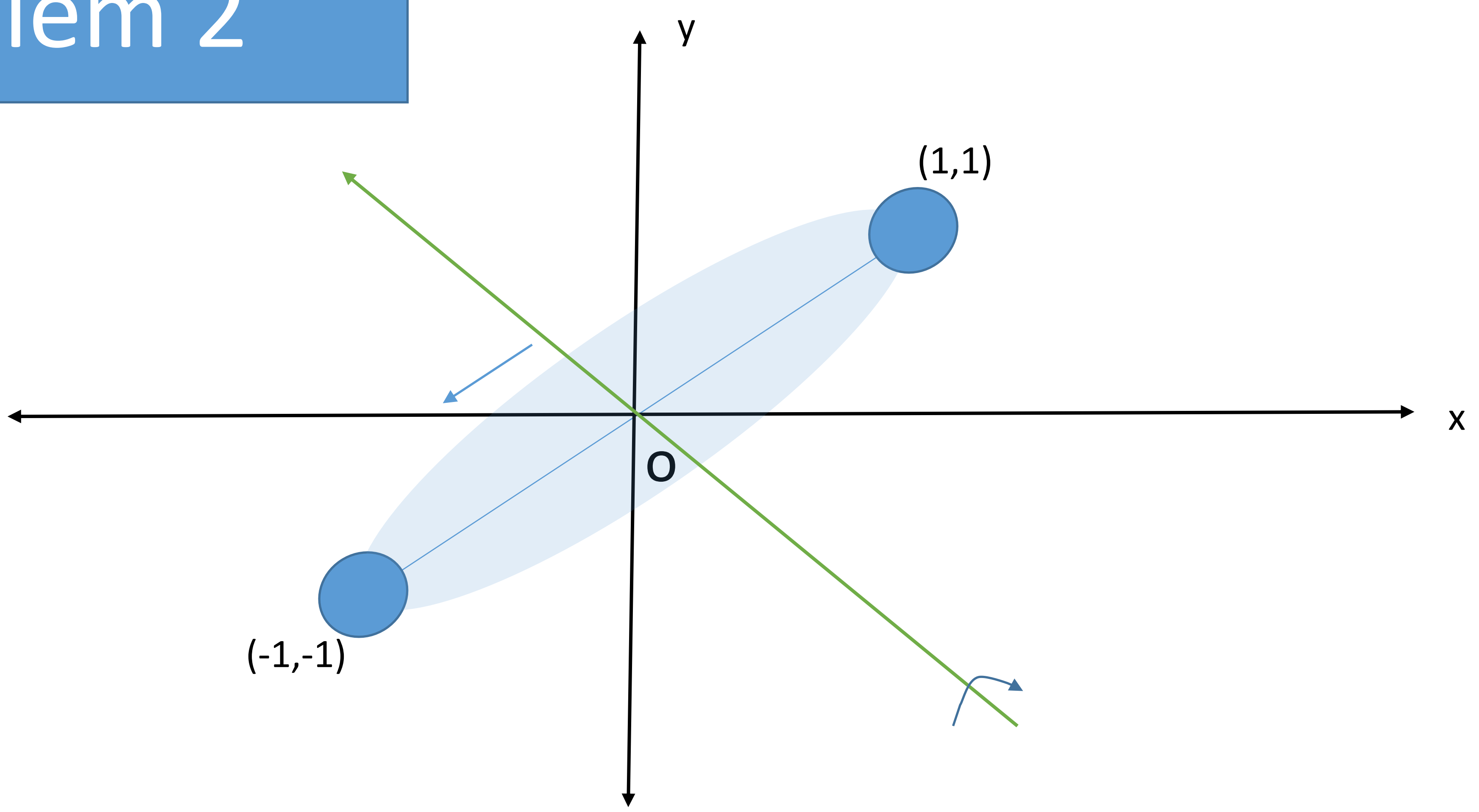
$$I_{xz}=-\sum m_j x_j z_j$$

For a continuous medium,

$$I_{xx}=\int\left(y^2+z^2\right)dm$$

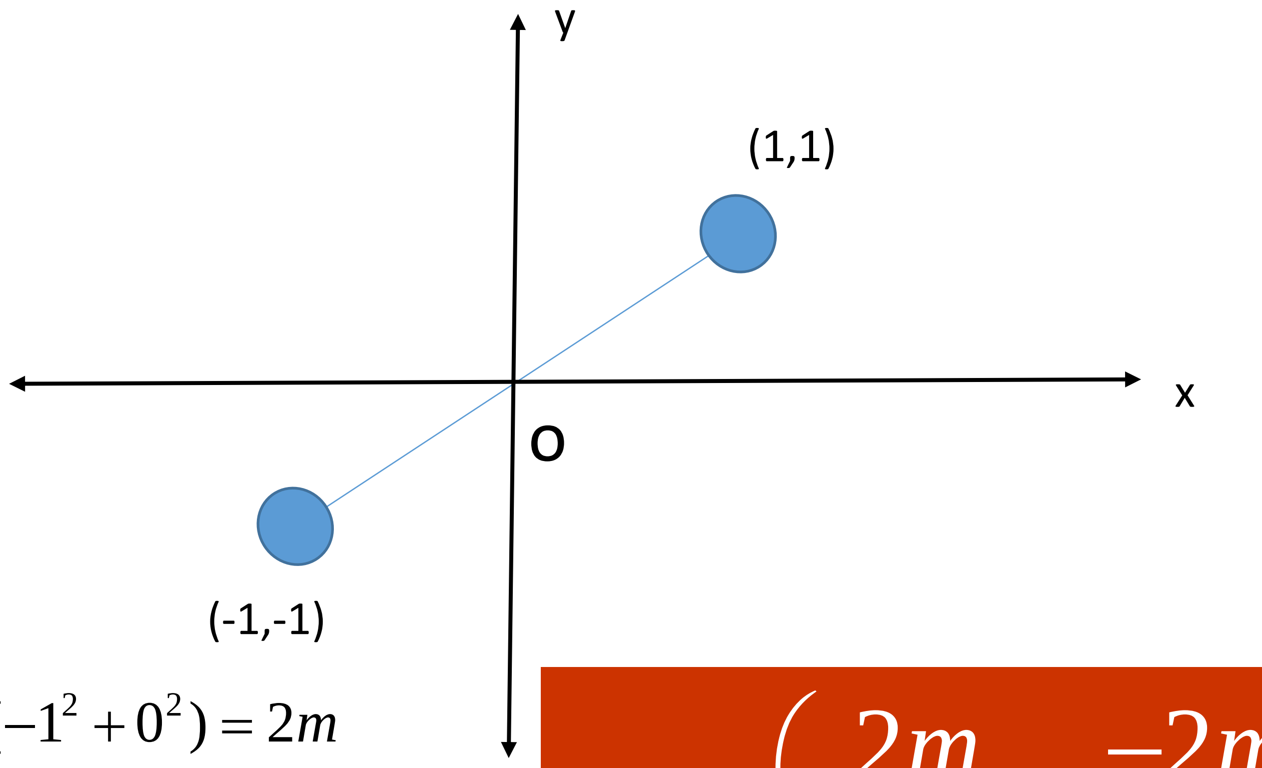
$$[I]=\begin{pmatrix} \int\left(y^2+z^2\right)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int\left(z^2+x^2\right)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int\left(x^2+y^2\right)dm \end{pmatrix}$$

Problem 2



Consider no mass for the interconnecting rod between the two balls. Also consider these balls as point particles.

a. **Find the Moment of Inertia Matrix**
b. **Angular momentum and torque matrix**
c. **Physically interpret the same for the current motion and if the system is rotated along z-axis**
d. **Find the net angular acceleration in this case also compare it, when the whole system is rotated in x-z plane**



$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2\right) \quad = m(1^2 + 0^2) + m(-1^2 + 0^2) = 2m$$

$$I_{yy} = \sum m_j \left(x_j^2 + z_j^2\right) \quad = m(1^2 + 0^2) + m(-1^2 + 0^2) = 2m$$

$$I_{zz} = \sum m_j \left(x_j^2 + y_j^2\right) \quad = m(1^2 + 1^2) + m(-1^2 + 1^2) = 4m$$

$$I_{xy} = -\sum m_j x_j y_j \quad = -m \times 1 \times 1 - m \times (-1) \times -1 = -2m$$

$$I_{yz} = -\sum m_j y_j z_j \quad = -m \times 1 \times 0 - m \times -1 \times 0 = 0$$

$$I_{xz} = -\sum m_j x_j z_j \quad = -m \times 1 \times 0 - m \times (-1) \times 0 = 0$$

$$[I] = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

Angular momentum and Torque

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

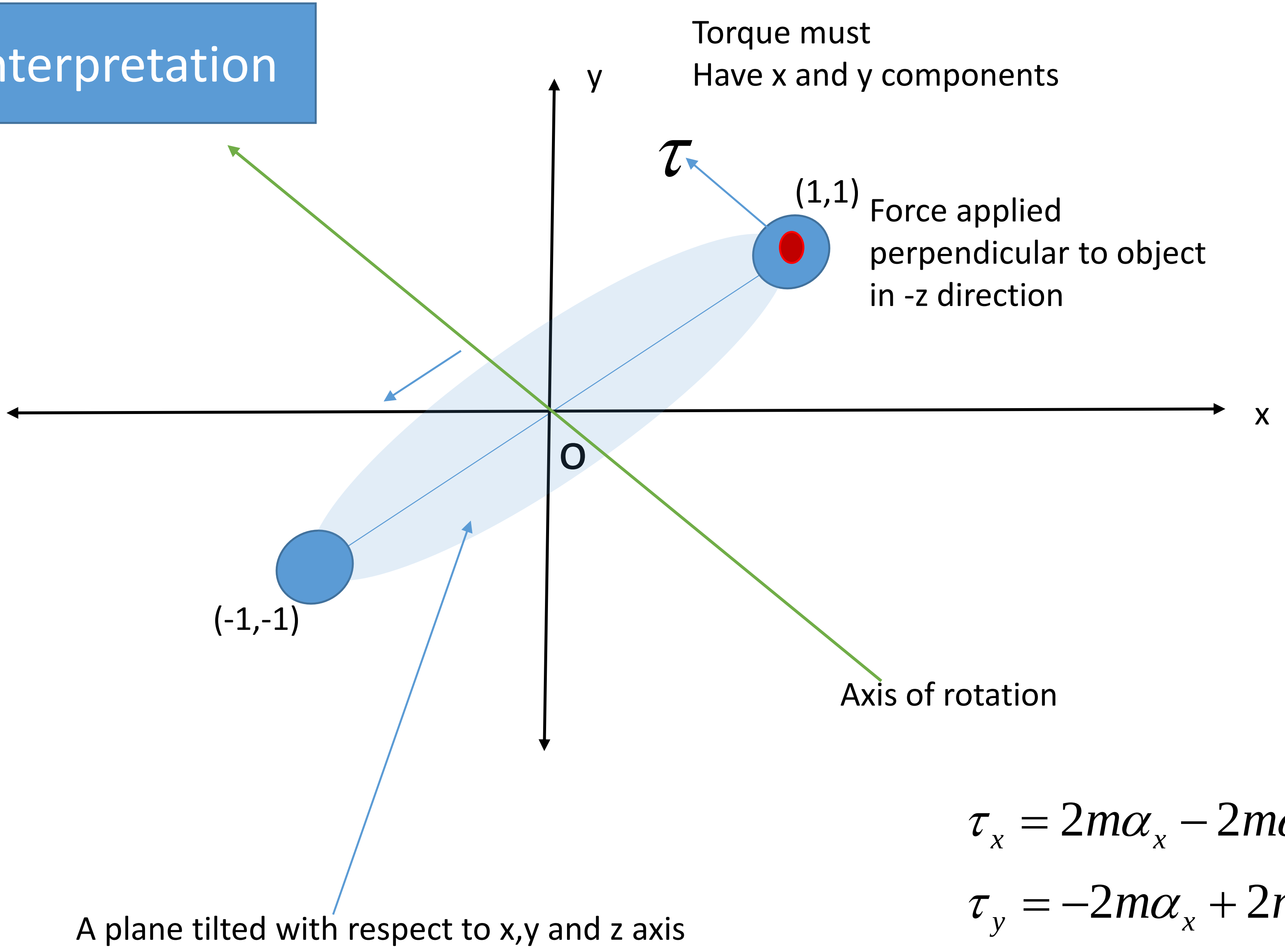
$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$\tau_x = 2m\alpha_x - 2m\alpha_y$$

$$\tau_y = -2m\alpha_x + 2m\alpha_y$$

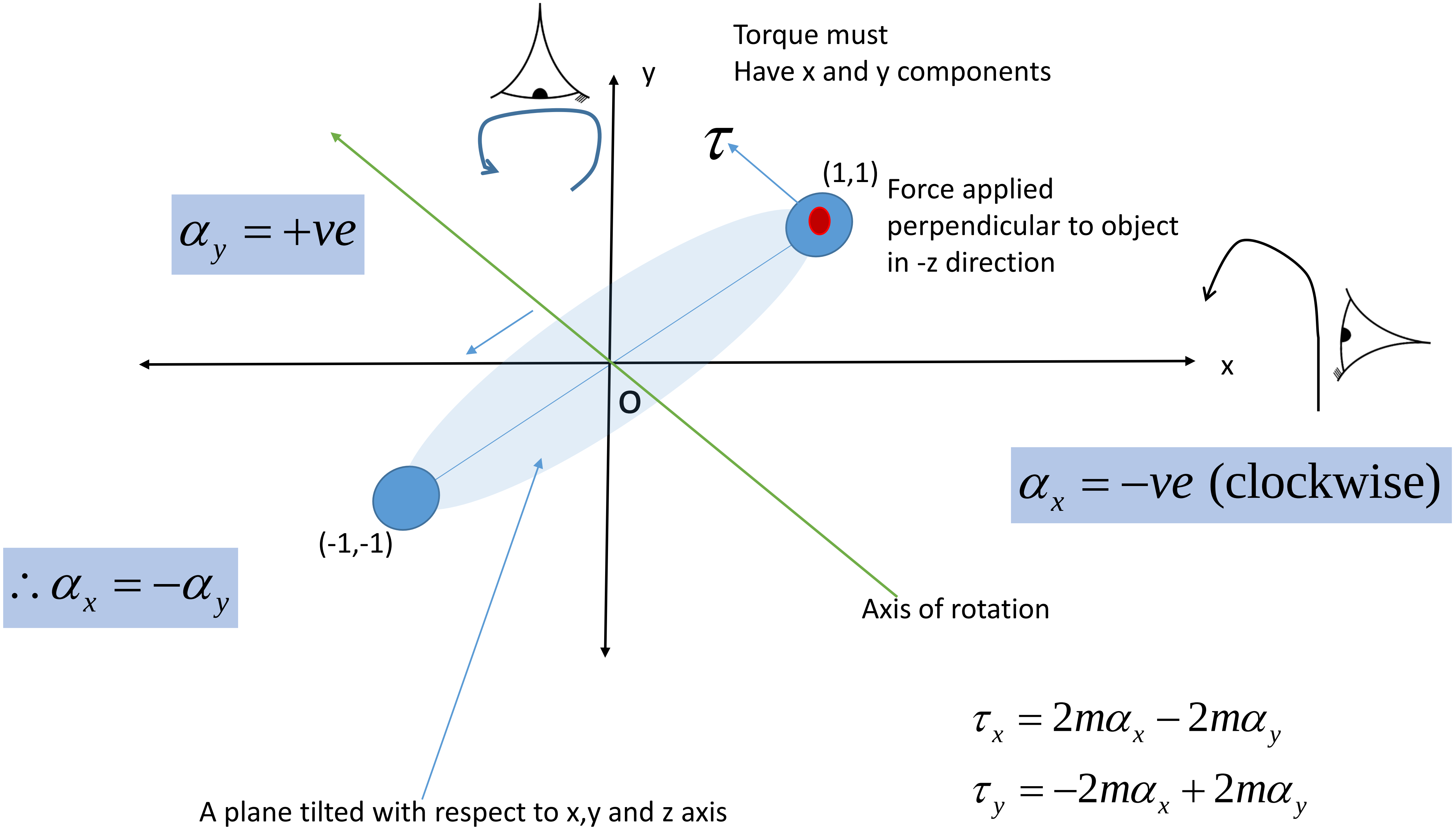
$$\tau_z = 4m\alpha_z$$

Physical Interpretation

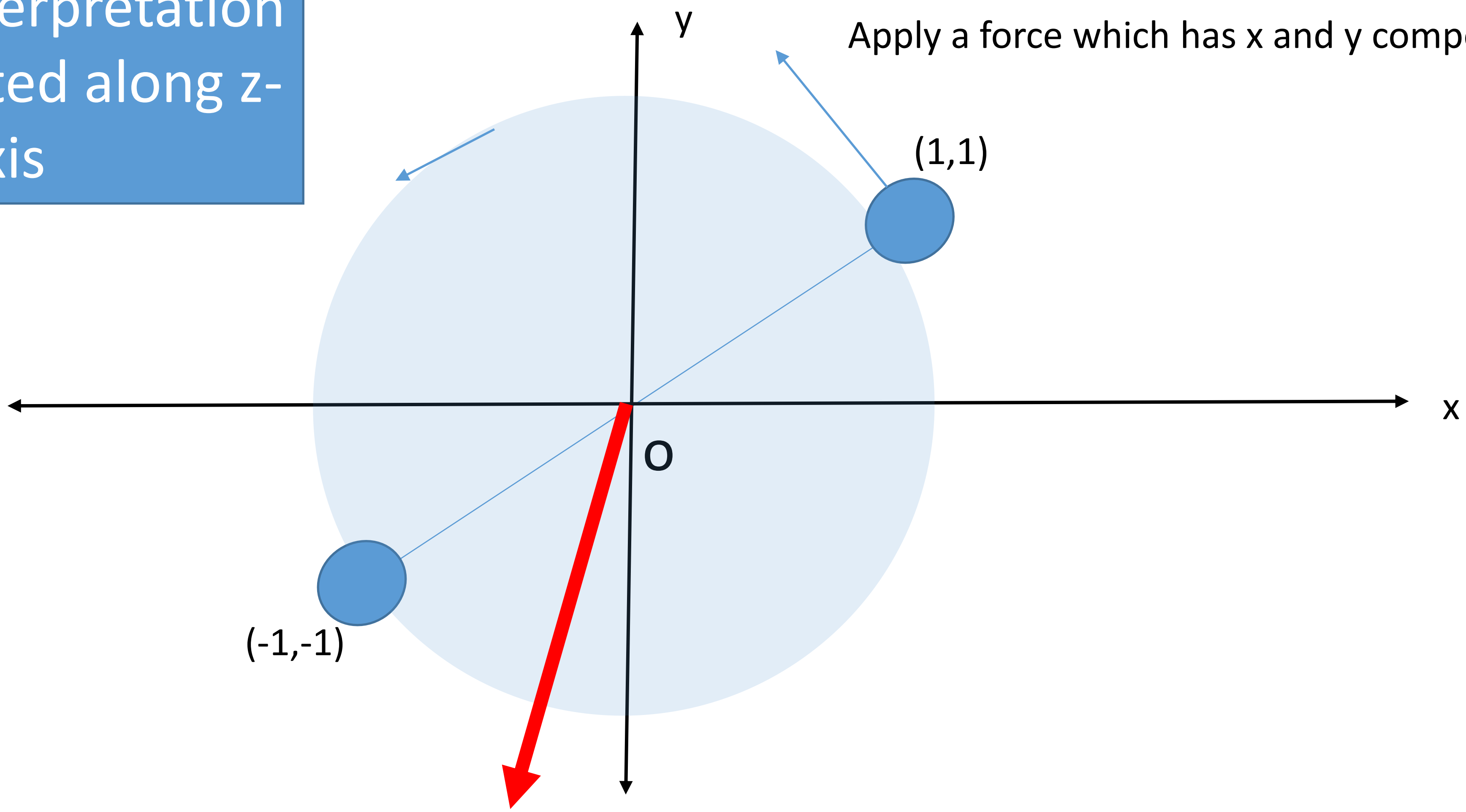


$$\tau_x = 2m\alpha_x - 2m\alpha_y$$

$$\tau_y = -2m\alpha_x + 2m\alpha_y$$


Torque must
Have x and y components
y
x
O
τ
(1,1)
(-1,-1)
Force applied
perpendicular to object
in -z direction
$\alpha_y = +ve$
$\alpha_x = -ve$ (clockwise)
$\therefore \alpha_x = -\alpha_y$
Axis of rotation
A plane tilted with respect to x,y and z axis
$\tau_x = 2m\alpha_x - 2m\alpha_y$
$\tau_y = -2m\alpha_x + 2m\alpha_y$

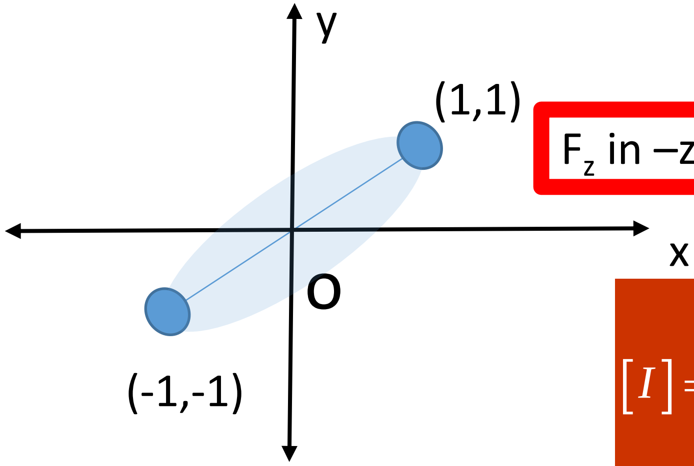
Physical Interpretation When rotated along z-axis



$$\tau_z = 4m\alpha_z$$

Find the net angular acceleration in this case also compare it, when the whole system is rotated in x-z plane

Angular acceleration



F_z in –z direction

$$[I]=\begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

$$\tau_x = 2m\alpha_x - 2m\alpha_y$$
$$\tau_y = -2m\alpha_x + 2m\alpha_y$$

$$\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} i & j & k \\ x & y & 0 \\ 0 & 0 & F_z \end{vmatrix} = F_z yi - F_z xk = \tau_x i + \tau_y j$$

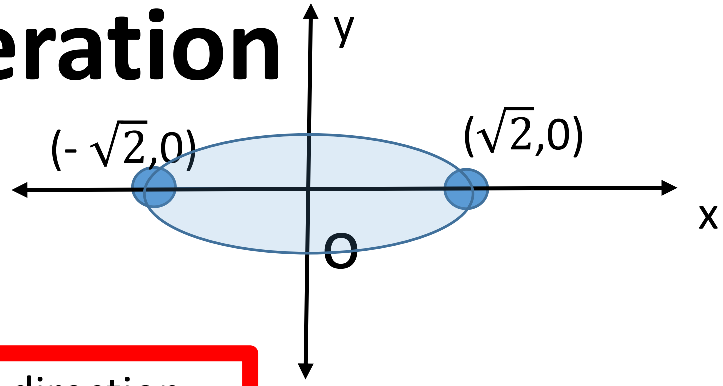
$$\alpha_x = -\alpha_y$$

$$\tau_x = F_z y = 2m\alpha_x - 2m\alpha_y$$
$$\tau_y = -F_z x = -2m\alpha_x + 2m\alpha_y$$

$$\alpha_y = -\frac{F_z y}{4m} \qquad \alpha_x = \frac{F_z y}{4m}$$

y=1

$$|\alpha| = \frac{F_z}{2\sqrt{2}m}$$



F_z in –z direction

$$[I]=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

$$\tau_y = 4m\alpha_y$$

$$xF_z = 4m\alpha_y$$

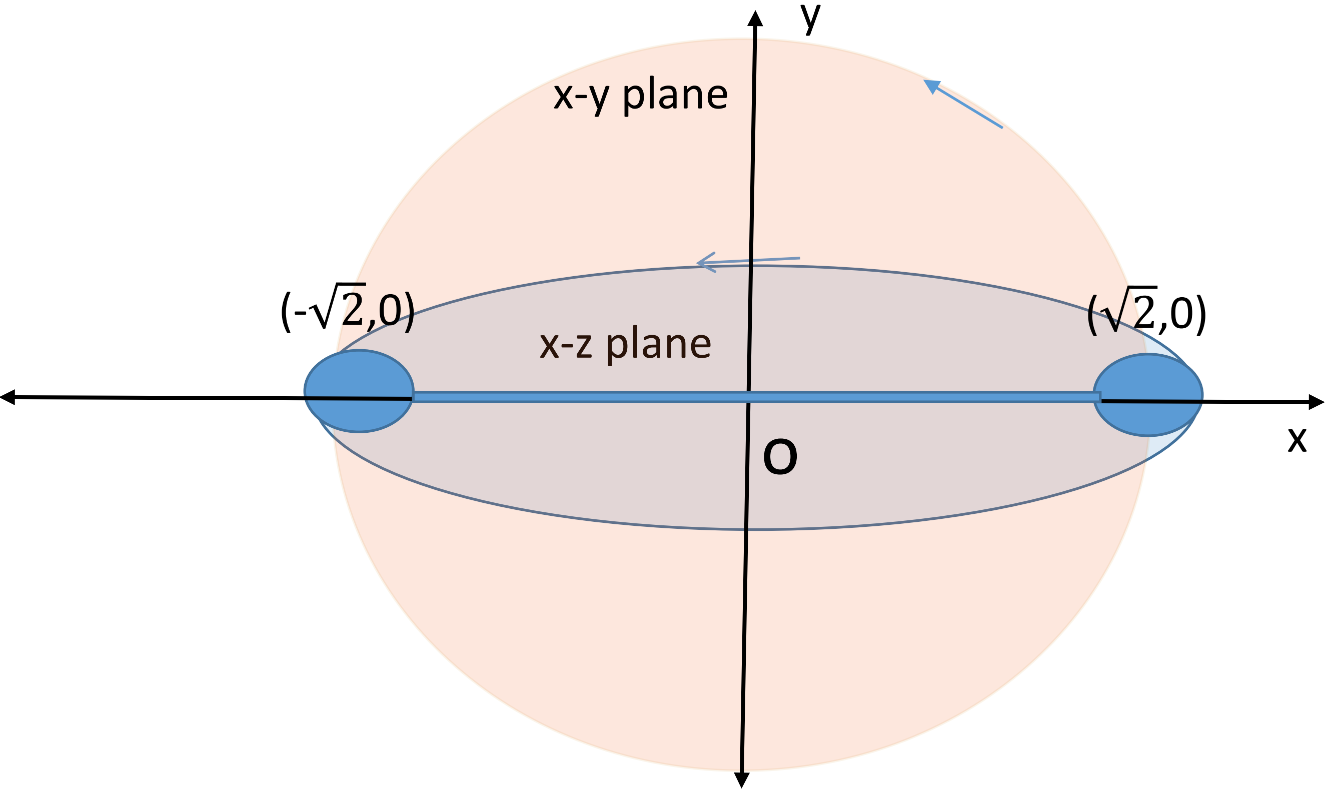
$$x = \sqrt{2}$$

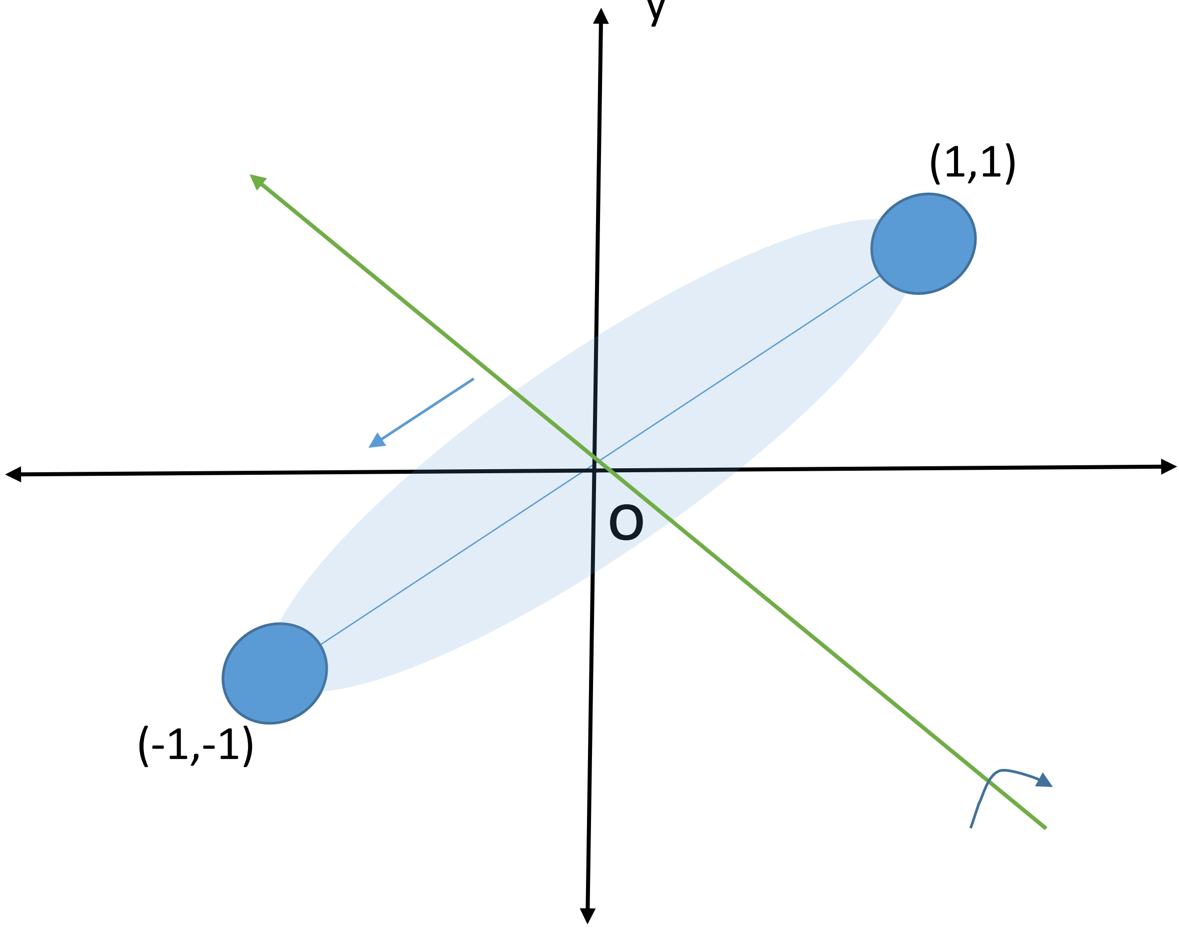
$$\alpha_y = \frac{xF_z}{4m}$$

$$|\alpha| = \frac{F_z}{2\sqrt{2}m}$$

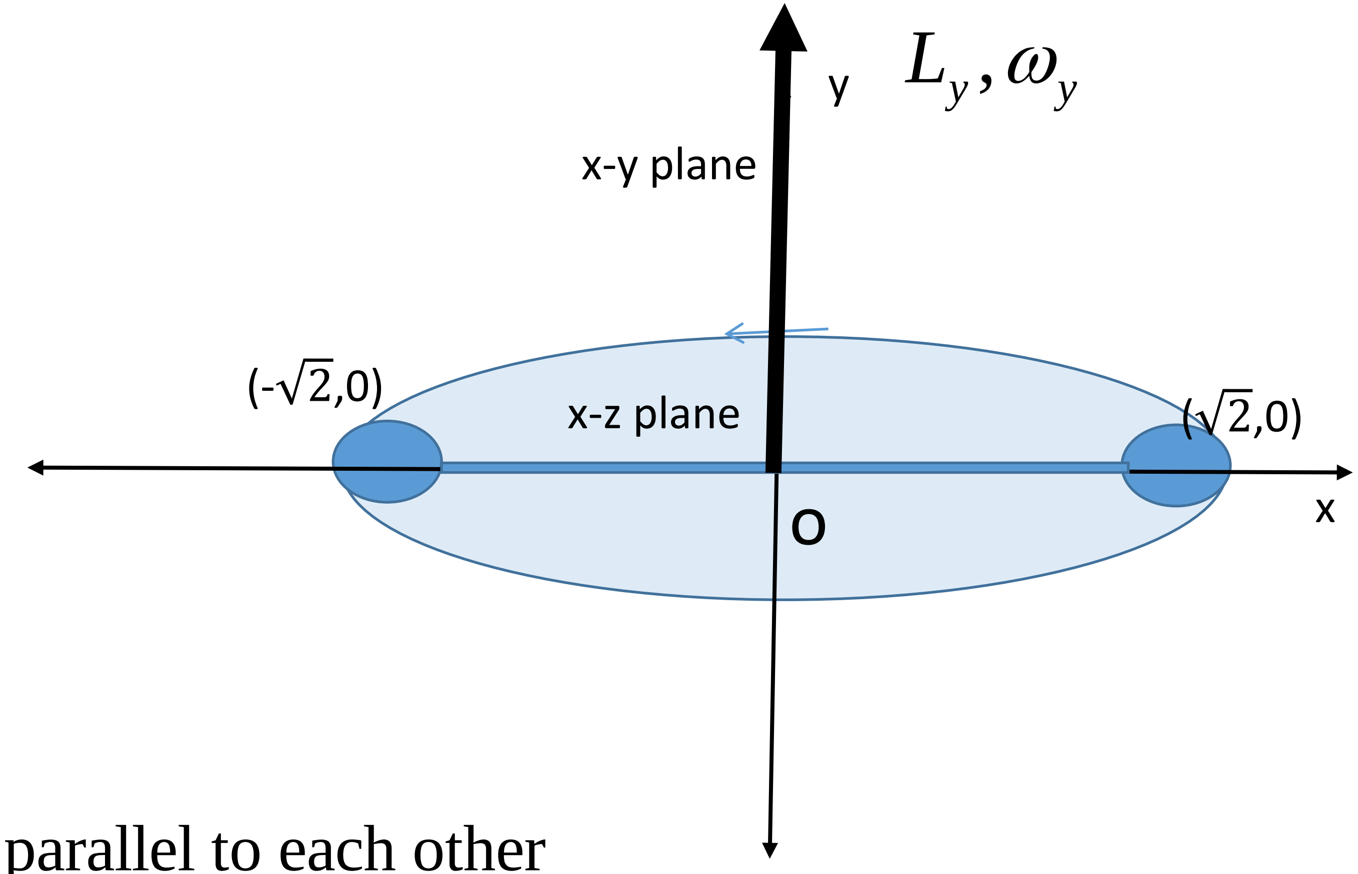
Problem 3

Find whether angular momentum and angular velocity are parallel in the below figure





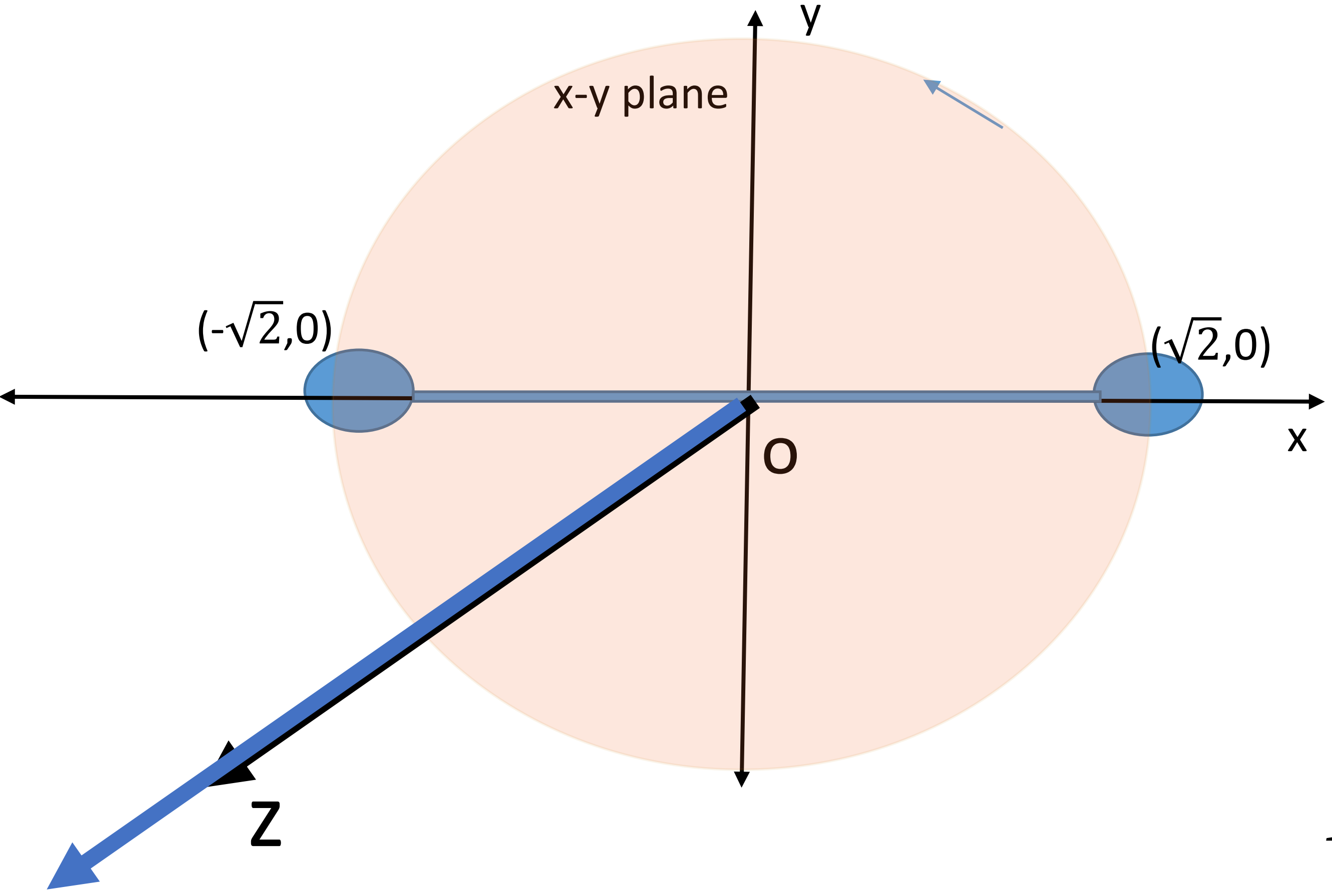
$$[I] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$



$$L_y = I_{yy}\omega_y = 4m\omega_y$$

$\vec{L}$ and $\vec{\omega}$ are parallel to each other

$\vec{L} = L_y \hat{e}_y$ and $\vec{\omega} = \omega_y \hat{e}_y$



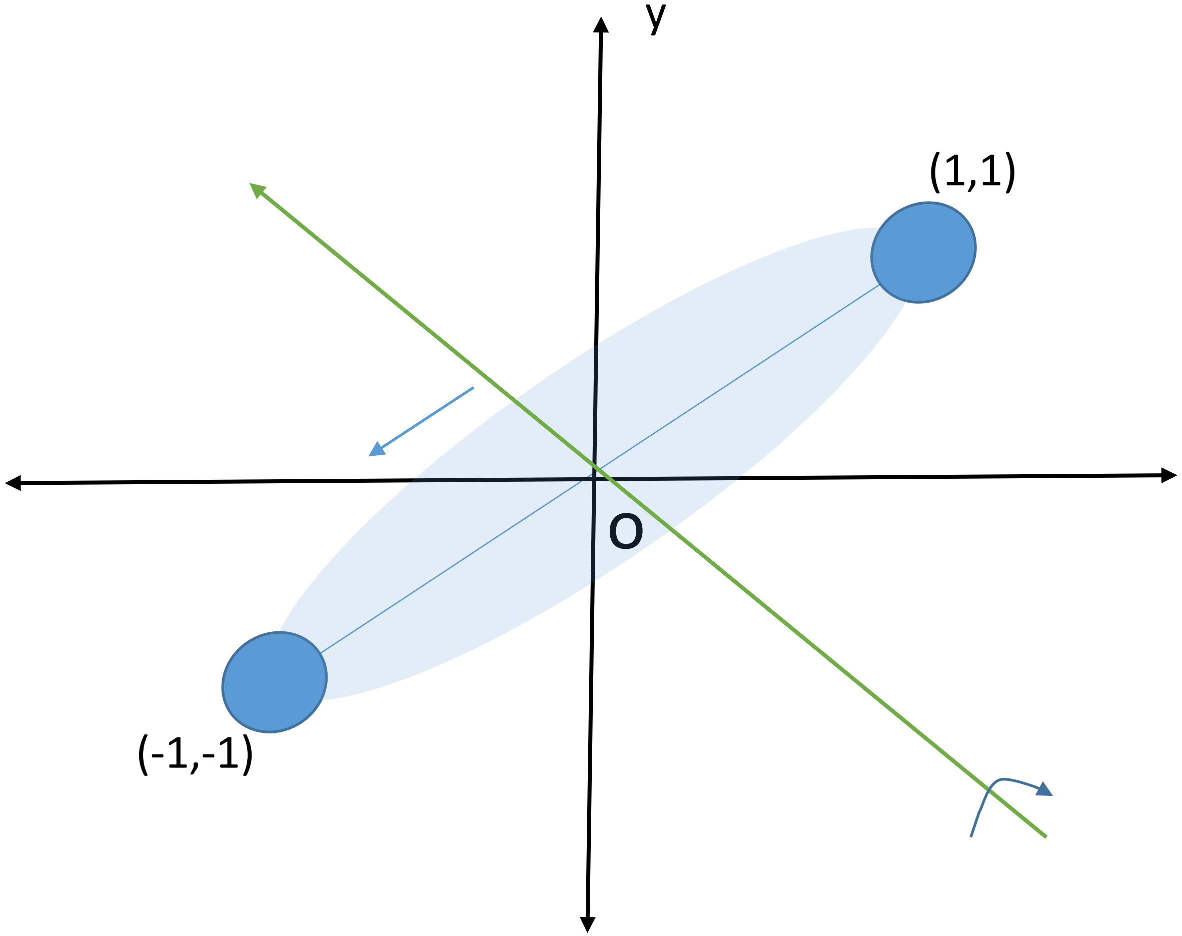
$$L_z, \omega_z$$

$$L_z = I_{zz}\omega_z = 4m\omega_z$$

$\vec{L}$ and $\vec{\omega}$ are parallel to each other

$\vec{L} = L_z\hat{e}_z$ and $\vec{\omega} = \omega\hat{e}_z$

Problem 2



$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ 0 \end{pmatrix}$$

$$L_x = 2m\omega_x - 2m\omega_y$$
$$L_y = -2m\omega_x + 2m\omega_y$$

$$L_x = 2m\omega_x - 2m\omega_y$$
$$L_y = -2m\omega_x + 2m\omega_y$$

$$\vec{L} = \vec{L}_x i + \vec{L}_y j$$

$$\vec{L} = (2m\omega_x - 2m\omega_y)i + (-2m\omega_x + 2m\omega_y)j$$

$$\vec{\omega} = \omega_x i + \omega_y j$$

What is the angle between $\vec{\mathrm{L}}$ and $\overrightarrow{\omega}$?

$$\cos\theta = \frac{\vec{L}.\vec{\omega}}{\left|\vec{L}\right|\left|\vec{\omega}\right|},$$ also we know $\omega_x = -\omega_y$

When $\omega_x = -\omega_y$

$$\vec{L} = -4\underline{m\omega_y} i + 4\underline{m\omega_y} j$$

$$\vec{\omega} = \underline{-\omega_y} i + \underline{\omega_y} j$$

$$\cos\theta = \frac{(-4m\omega_y i + 4m\omega_y j).(-\omega_y i + \omega_y j)}{\sqrt{32m^2\omega_y{}^2}\sqrt{2\omega_y{}^2}} = \frac{8m\omega_y{}^2}{\sqrt{64m^2\omega_y{}^2}} = 1$$

$\theta = 0 \rightarrow \vec{L}$ and $\vec{\omega}$ are parallel

$\frac{L_x}{L_y} = \frac{\omega_x}{\omega_y}$ **Angular momentum and Angular velocity have same slopes**

$$\vec{L} = \left(-4m\omega_y, 4m\omega_y, 0\right) \text{ and } \vec{\omega} = \left(-\omega_y, \omega_y, 0\right)$$

In the matrix form we may write

$$\vec{L} = [I][\vec{\omega}]$$

$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ 0 \end{pmatrix}$$

$$\omega_x = -\omega_y$$

$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} -\omega_y \\ \omega_y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} -\omega_y \\ \omega_y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} -\omega_y \\ \omega_y \\ 0 \end{pmatrix} = \begin{bmatrix} -4m\omega_y \\ 4m\omega_y \\ 0 \end{bmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = 4m \begin{pmatrix} -\omega_y \\ \omega_y \\ 0 \end{pmatrix}$$

$$\vec{L} = [I][\vec{\omega}] = 4m[\vec{\omega}]$$

If $\vec{L}$ and $\vec{\omega}$ are parallel, one can write $\vec{L} = [I][\vec{\omega}] = \lambda[\vec{\omega}]$

Problem 4 Rotation of a square plate

Consider rotation of a square plate of side a and mass M about an axis in the plane of the plate and making an angle α with the x-axis.

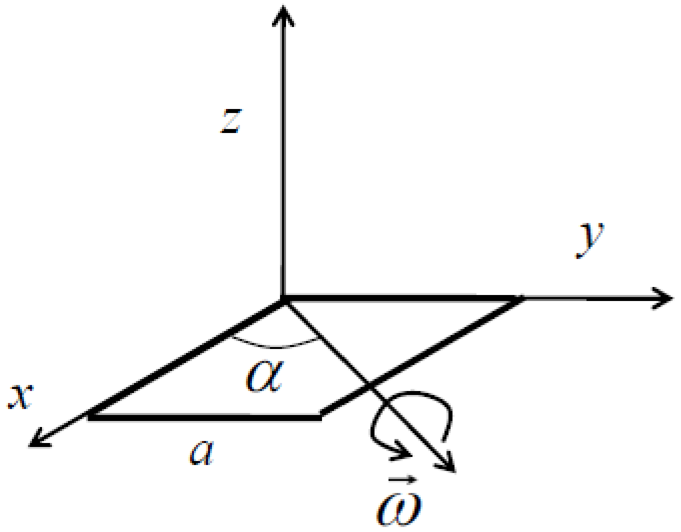
(a) Write the matrix for angular momentum **L**

(b) For what angle **L** and **ω** becomes parallel?

(c) For square plate when the moment of inertia tensor becomes diagonal?

Also surface mass density is defined as $\sigma = \frac{M}{A}$.

M is the mass of the plate and A is the area



Rotation of a square plate

(a)angular momentum **L**

$$\vec{L} = [I][\vec{\omega}]$$

$$[I] = \begin{pmatrix} \int (y^2+z^2)dm & -\int xydm & -\int zxdm \\ -\int xydm & \int (z^2+x^2)dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int (x^2+y^2)dm \end{pmatrix}$$

$$\vec{\omega} = \begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix}$$

$$M = \sigma A \rightarrow \sigma = \frac{M}{a^2};$$

$$\Delta M = \sigma \Delta x \Delta y$$

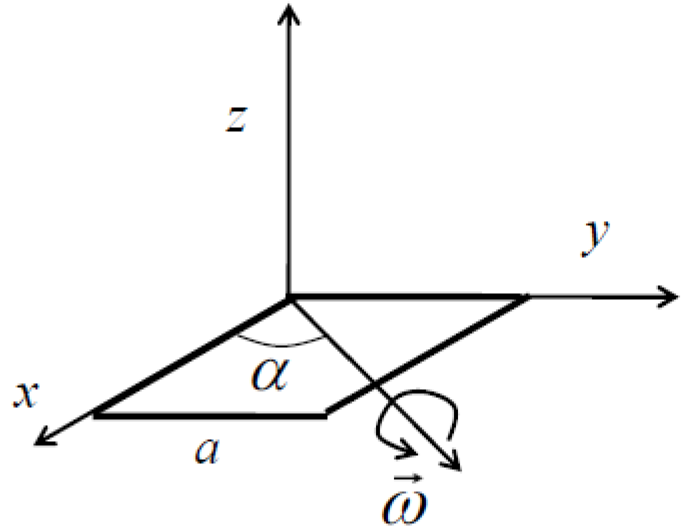
$$I_{xx} = \iint (y^2+z^2)dm = \int_0^a\int_0^a \sigma(y^2+z^2)dxdy$$

$$I_{xx} = \int_{x=0}^{a}\int_{y=0}^{a} \sigma y^2 dxdy = \frac{1}{3}\sigma a^4 = \frac{1}{3}\times\frac{M}{a^2}\times a^4 = \frac{1}{3}Ma^2$$

$$I_{yy} = \iint (z^2+x^2)dm = \int_0^a\int_0^a \sigma(x^2)dxdy = \frac{\sigma a^4}{3} = \frac{1}{3}Ma^2$$

$$I_{zz} = \iint (x^2+y^2)dm = \int_0^a\int_0^a \sigma(x^2+y^2)dxdy = \frac{2\sigma a^4}{3} = \frac{2Ma^2}{3}$$

$$I_{xy} = -\int_{x=0}^{a}\int_{y=0}^{a} \sigma xydxdy = -\frac{1}{4}\sigma a^4 = -\frac{1}{4}Ma^2$$



Rotation of a square plate

(a)angular moment L

$$\vec{L} = [I][\vec{\omega}]$$

$$\vec{\omega} = \begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix}$$

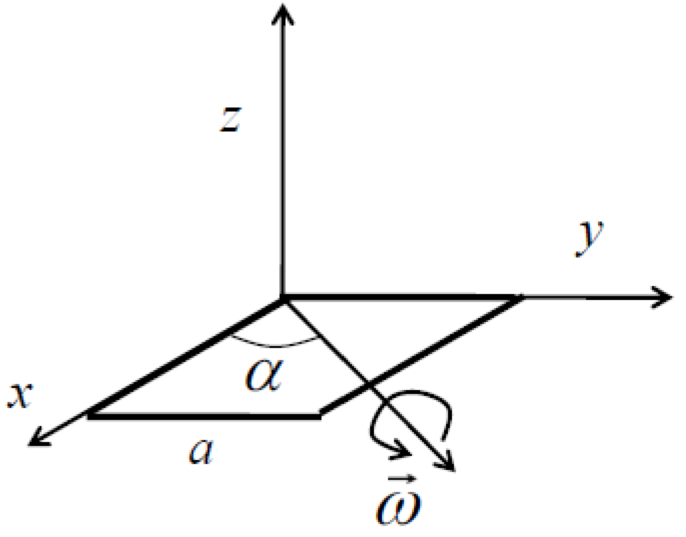
$$\vec{L} = \begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix} \begin{pmatrix} \omega\cos\alpha \\ \omega\sin\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} Ma^2\omega\left(\frac{1}{3}\cos\alpha - \frac{1}{4}\sin\alpha\right) \\ Ma^2\omega\left(-\frac{1}{4}\cos\alpha + \frac{1}{3}\sin\alpha\right) \\ 0 \end{pmatrix}$$

Is $\vec{L}$ and $\vec{\omega}$ parallel?

$$\vec{L} = \left(Ma^2\omega\left(\frac{1}{3}\cos\alpha - \frac{1}{4}\sin\alpha\right), Ma^2\omega\left(-\frac{1}{4}\cos\alpha + \frac{1}{3}\sin\alpha\right), 0 \right) \text{ and } \vec{\omega} = (\omega\cos\alpha, \omega\sin\alpha, 0)$$

$$\frac{L_x}{L_y} \neq \frac{\omega_x}{\omega_y}$$

$\vec{L}$ and $\vec{\omega}$ are not parallel



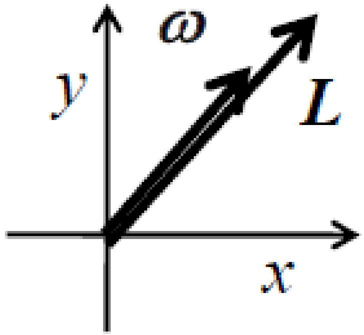
Rotation of a square plate

(b) For what angle L and ω becomes parallel?

For $\alpha = 45^\circ$,

$$\vec{L} = \left(\frac{1}{12\sqrt{2}} Ma^2\omega, \frac{1}{12\sqrt{2}} Ma^2\omega, 0 \right) \text{ and } \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\frac{L_x}{L_y} = \frac{\omega_x}{\omega_y}$$



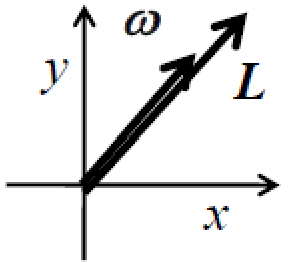
Rotation of a square plate

(b) For what angle L and ω becomes parallel?

For $\alpha = 45^{\circ}$,

$$\vec{L} = \left(\frac{1}{12\sqrt{2}} Ma^2\omega, \frac{1}{12\sqrt{2}} Ma^2\omega, 0 \right) \quad \text{and} \quad \vec{\omega} = \begin{pmatrix} \omega/\sqrt{2} \\ \omega/\sqrt{2} \\ 0 \end{pmatrix}$$

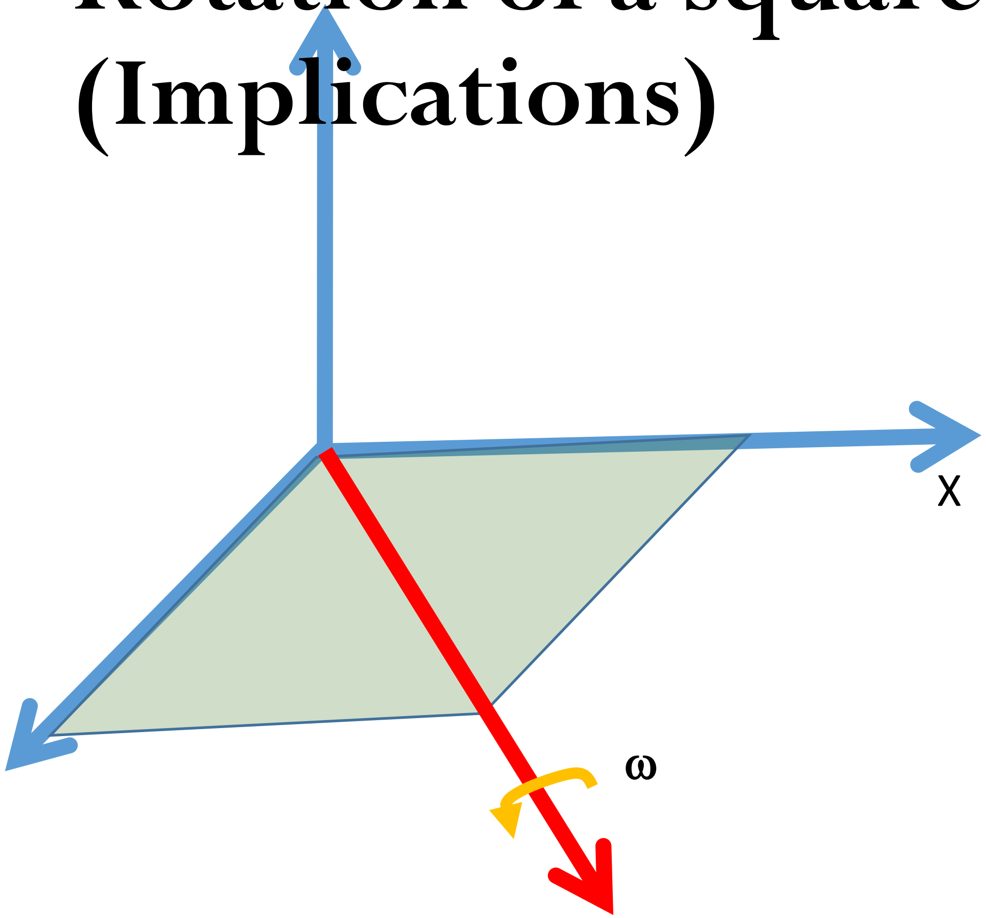
$$\begin{pmatrix} L_x \\ L_y \\ 0 \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ \frac{\omega}{\sqrt{2}} \\ 0 \end{pmatrix}$$



$$\vec{L} = [I][\vec{\omega}] = \lambda \vec{\omega}$$

$$\lambda = \frac{Ma^2}{12}$$

Rotation of a square plate (Implications)

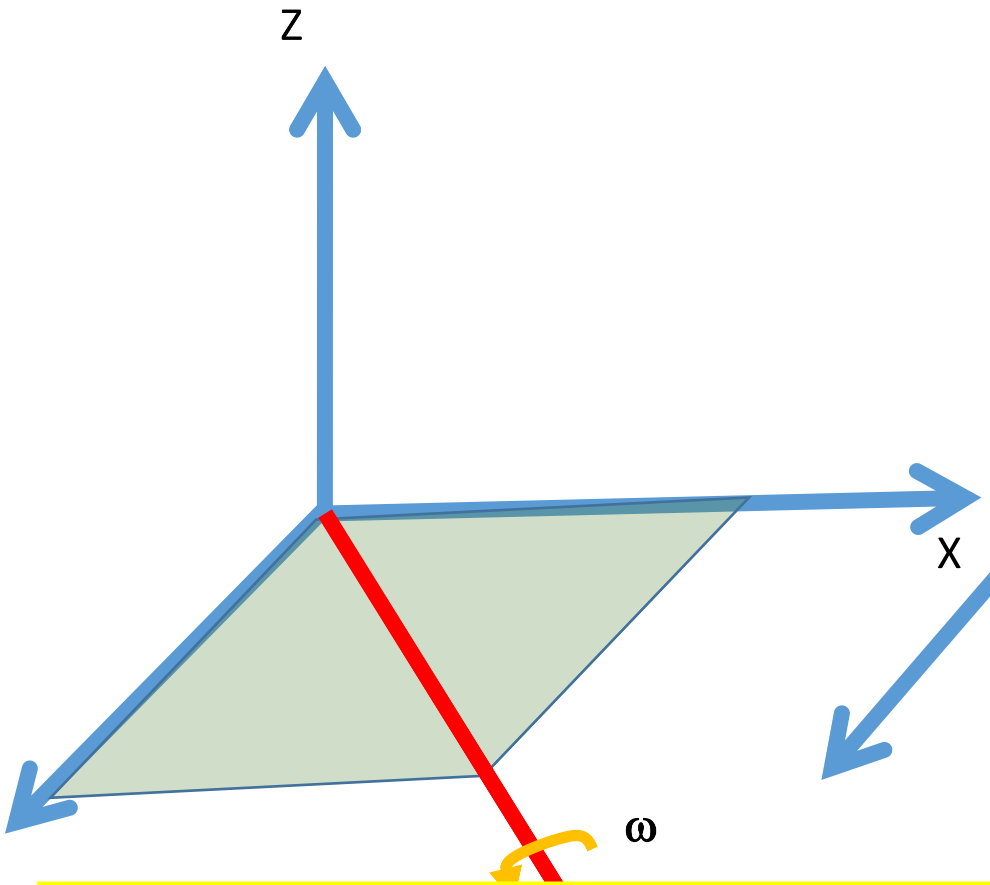


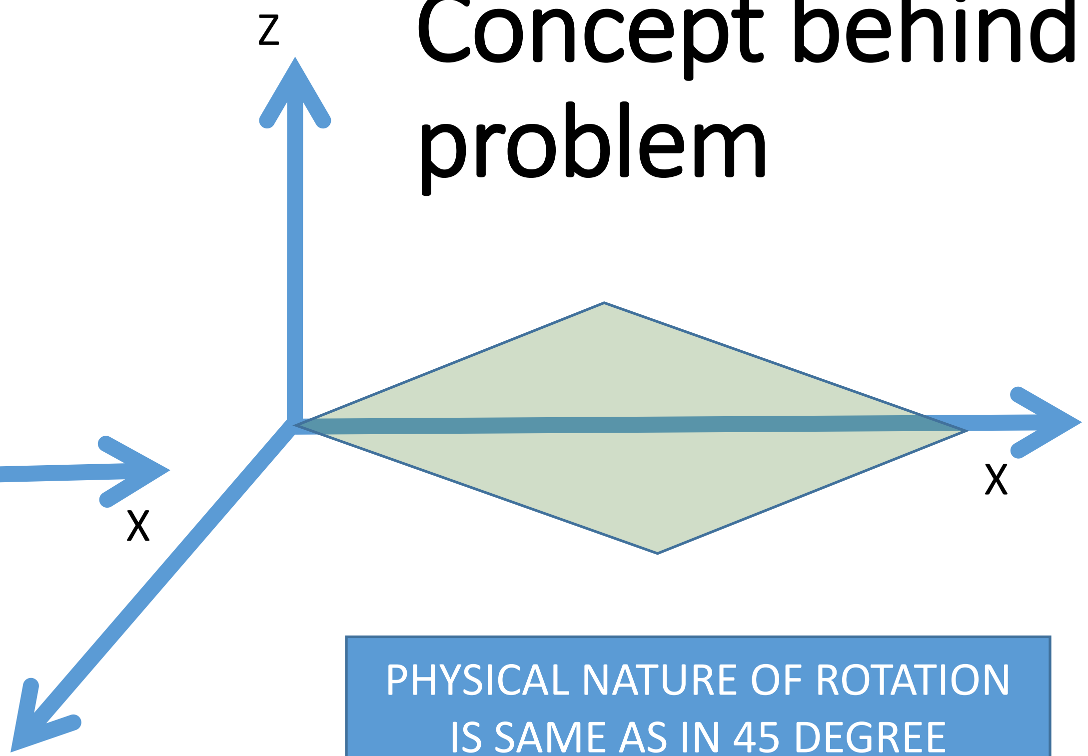
$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$\vec{L} = [I][\vec{\omega}]$$

L and **ω** are parallel!

Concept behind the problem





PHYSICAL NATURE OF ROTATION IS SAME AS IN 45 DEGREE ROTATION

Use of symmetry will ensure diagonal Moment of Inertia tensor

$$[I] = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

$$L_x = I_{xx}\omega_x$$

$$\tau_x = I_{xx}\alpha_x$$

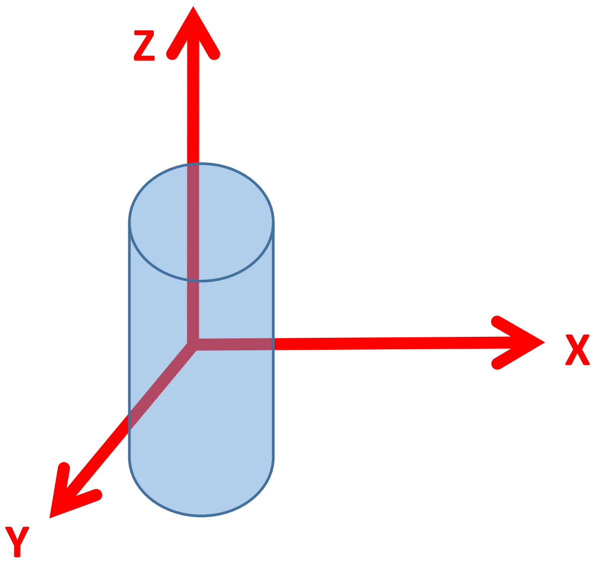
Principal Axis

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Cumbersome!

Principal axes are the orthogonal axes for Which [I] is diagonal

$$[I] = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$



$$L_x = I_{xx}\omega_x \quad L_y = I_{yy}\omega_y$$

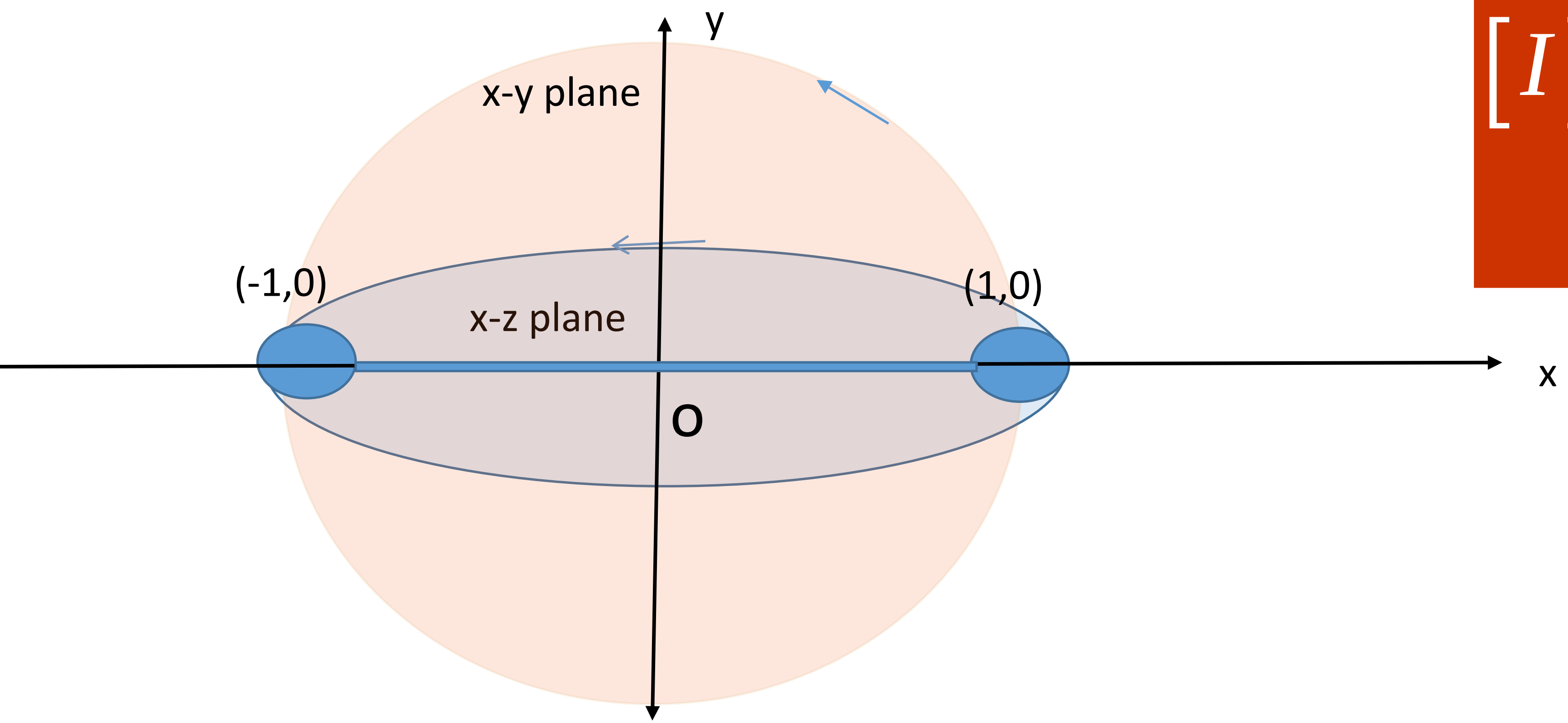
$$L_z = I_{zz}\omega_z$$

Note!

Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the principal axes.

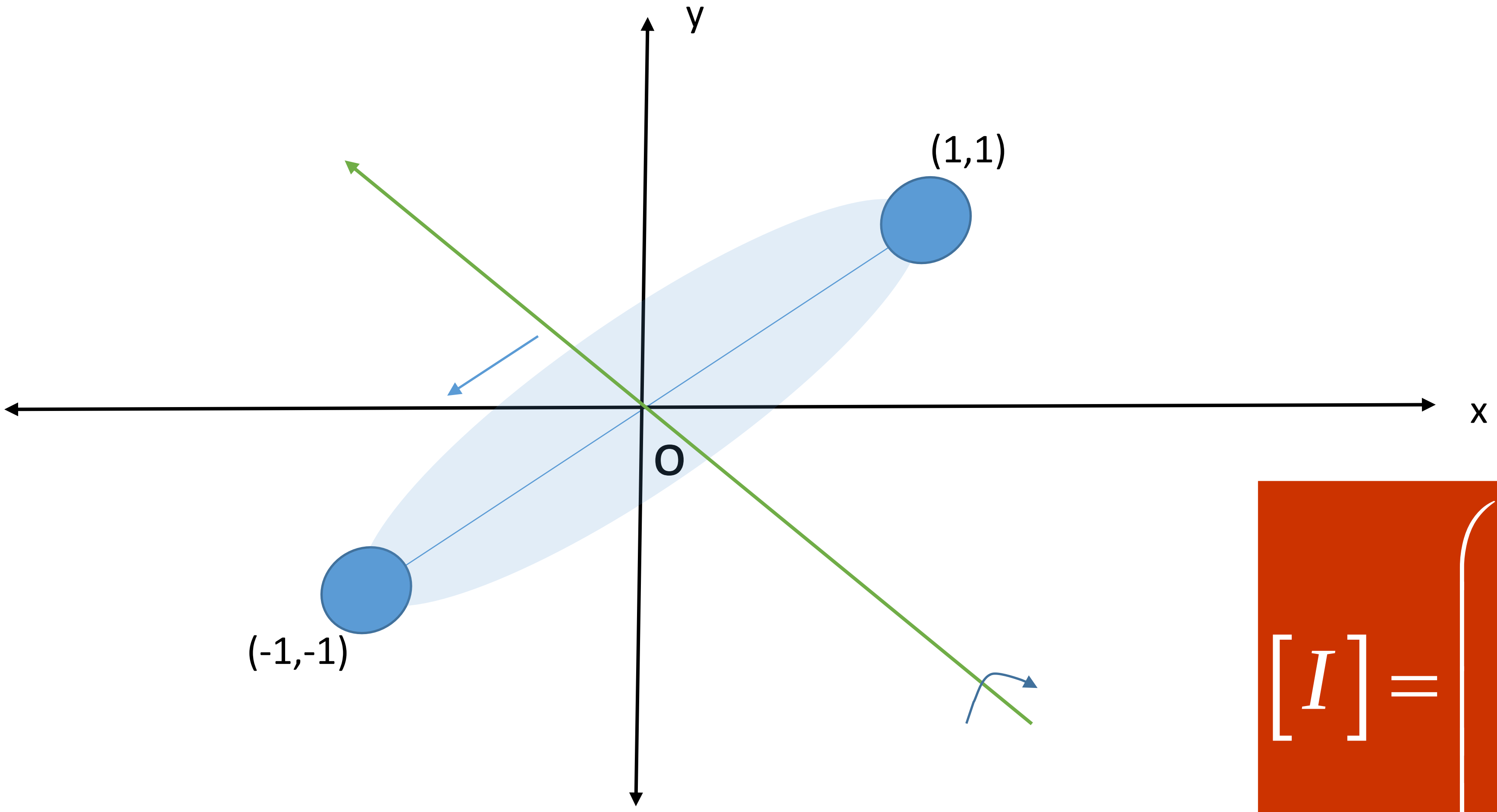
Let's recollect the problems we did………

Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the <u>principal axes</u>



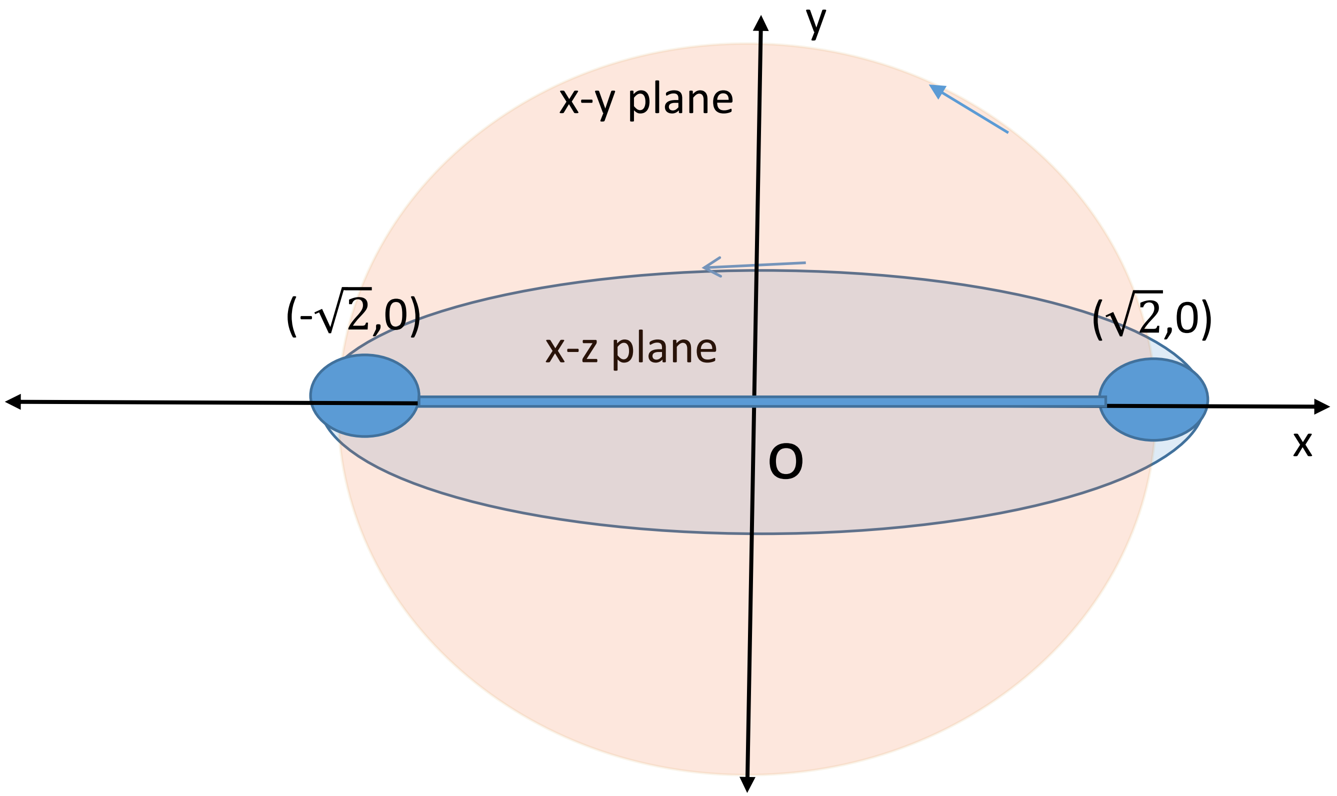
$$[I]=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix}$$

Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the principal axes



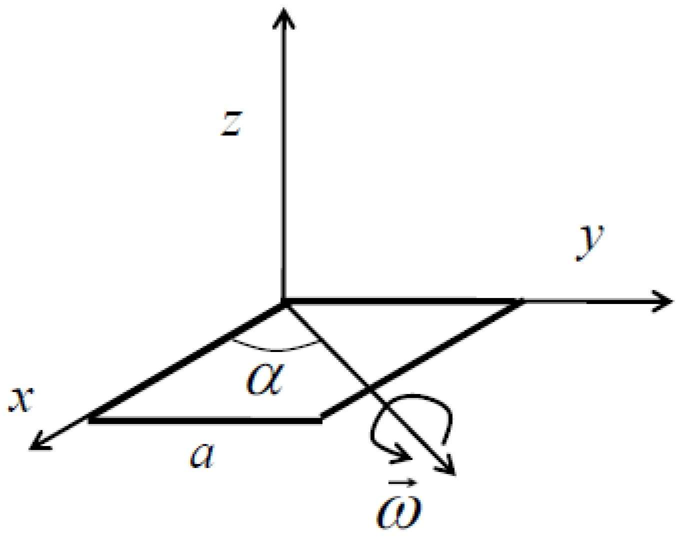
$$[I] = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

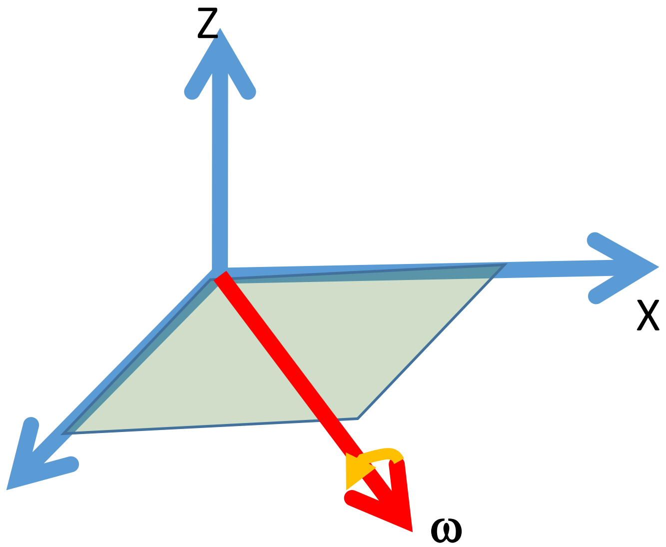
Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the <u>principal axes</u>

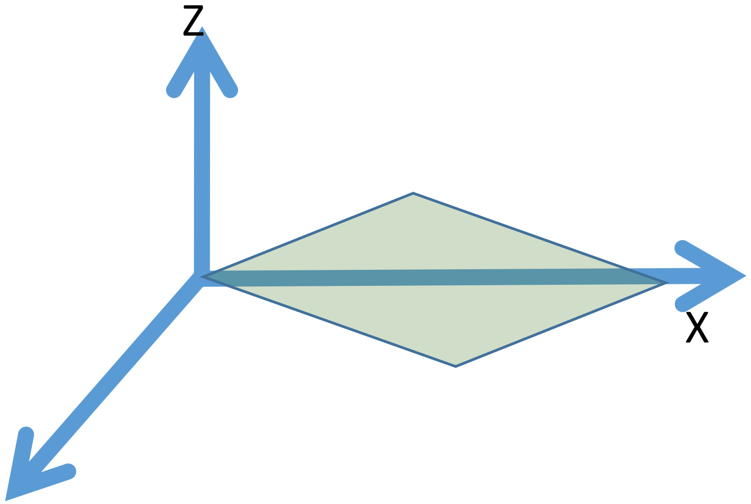


$$[I]=\begin{pmatrix}0 & 0 & 0\\ 0 & 4m & 0\\ 0 & 0 & 4m\end{pmatrix}$$

Whenever ω is parallel to L, choosing the corresponding direction of rotation as co-ordinate axis will be the principal axes







$$[I] = \begin{pmatrix} Ma^2/3 & -Ma^2/4 & 0 \\ -Ma^2/4 & Ma^2/3 & 0 \\ 0 & 0 & 2Ma^2/3 \end{pmatrix}$$

$$[I] = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

Quiz 2: 1st May 2022
Mode: Descriptive, Online (Teams)
Time: 9.00 AM
Duration: 20 minutes
Syllabus: Chapter 3 (Work energy and conservation laws)