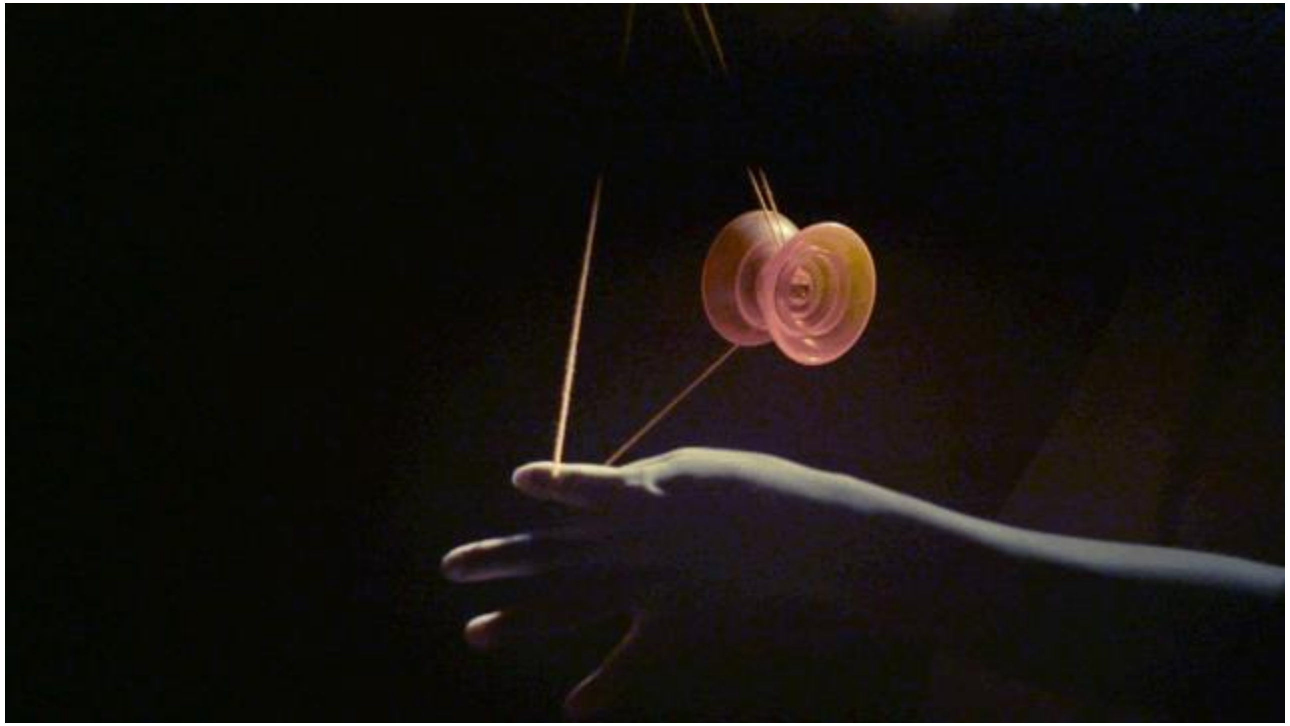
Chapter 4

RIGID BODY IN MOTION



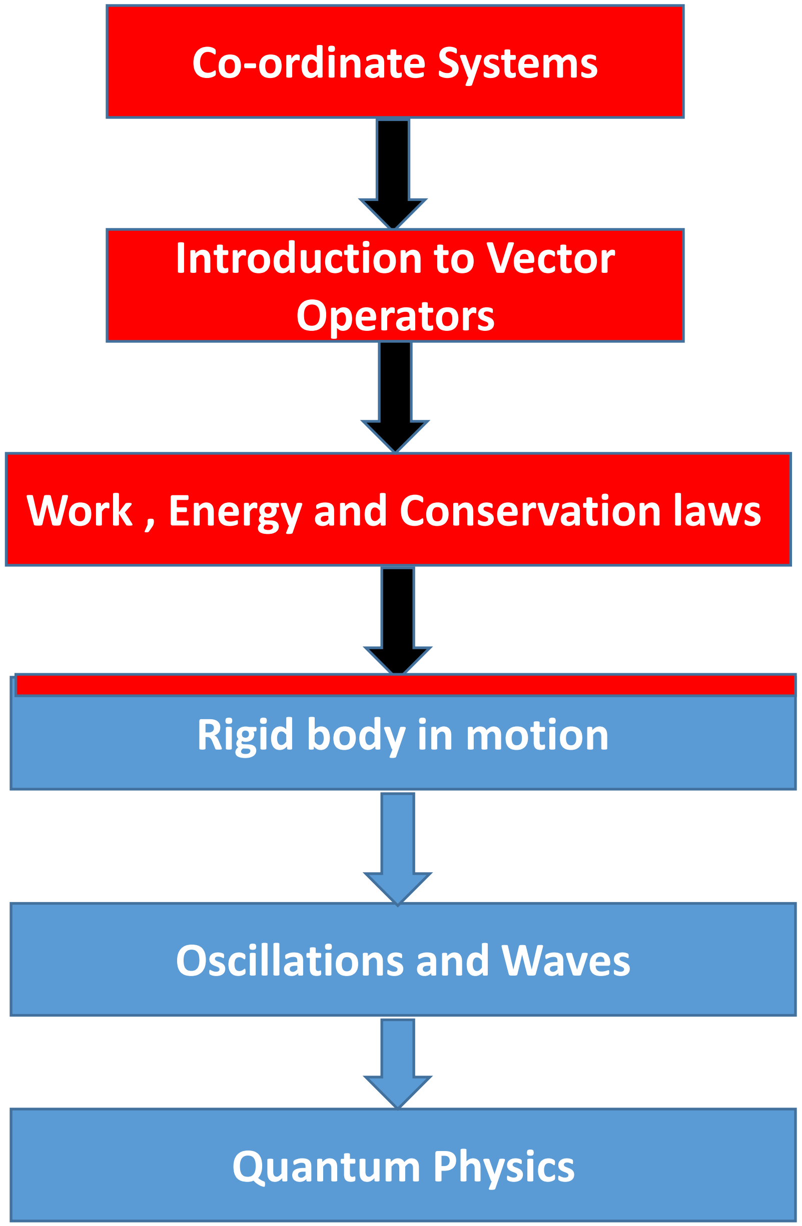
Quiz 2: 1st May 2022

Mode: Descriptive, Online (Teams)

Time: 10.00 AM

Syllabus: Chapter 3 (Work energy and conservation laws)

Highlights of the course



Angular momentum of a rigid body

RECAP

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

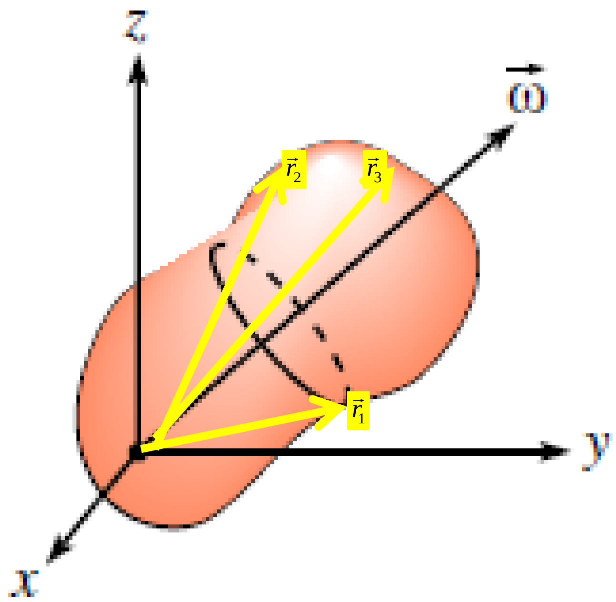
$$\vec{L} = \vec{r} \times m\vec{v}$$

$$\vec{L} = [I][\vec{\omega}]$$

$$\vec{\tau} = [I][\vec{\alpha}]$$

Will these relations hold good in the case of Rigid body rotation?

Angular momentum of a rigid body



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} = m\frac{d\vec{r}}{dt} = m(\vec{\omega} \times \vec{r})$$

$$\vec{L} = \sum_{j=1}^{N} \left[\vec{r}_j \times m_j \left(\vec{\omega} \times \vec{r}_j \right) \right]$$

$$\vec{\omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z$$

$$\vec{\omega} \times \vec{r}_j = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \omega_x & \omega_y & \omega_z \\ x_j & y_j & z_j \end{vmatrix}$$

$$\vec{r}_j \times m_j \left(\vec{\omega} \times \vec{r}_j \right) ?$$

$$\vec{r}_j \times m_j\left(\vec{\omega}\times\vec{r}_j\right)?$$

$$\vec{\omega}\times\vec{r}_j = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \omega_x & \omega_y & \omega_z \\ x_j & y_j & z_j \end{vmatrix}$$

$$m_j\vec{\omega}\times\vec{r}_j = m_j(\omega_y z_j - y_j\omega_z)\hat{e}_x - m_j(\omega_x z_j - x_j\omega_z)\hat{e}_y + m_j(\omega_x y_j - x_j\omega_y)\hat{e}_z$$

$$\vec{r}_j \times m_j\left(\vec{\omega}\times\vec{r}_j\right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ x_j & y_j & z_j \\ m_j(\omega_y z_j - y_j\omega_z) & -m_j(\omega_x z_j - x_j\omega_z) & m_j(\omega_x y_j - x_j\omega_y) \end{vmatrix}$$

$$L_x = m_j\left(y_j^2 + z_j^2\right)\omega_x - m_j x_j y_j \omega_y - m_j x_j z_j \omega_z$$

Angular momentum of a rigid body

$$\vec{L} = \sum_{j=1}^{N} \left[\vec{r}_j \times m_j \left(\vec{\omega} \times \vec{r}_j \right) \right]$$

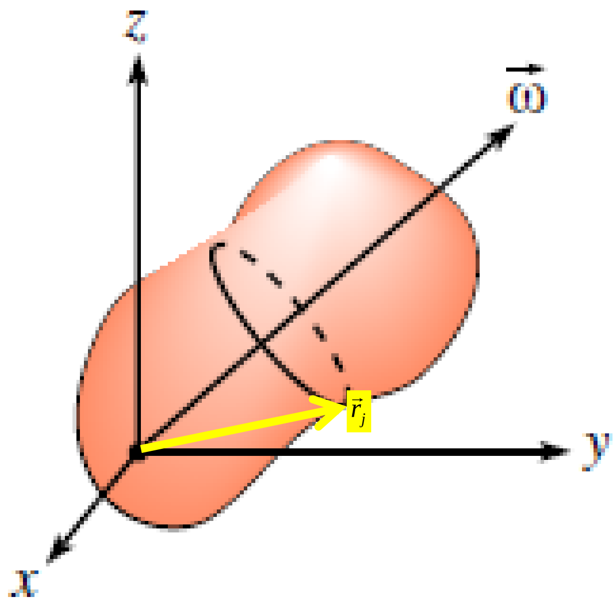
$$L_x = \sum_{j=1}^{N} \left[m_j \left(y_j^2 + z_j^2 \right) \omega_x - m_j x_j y_j \omega_y - m_j x_j z_j \omega_z \right]$$

Let us introduce moment of Inertias

$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2 \right) \qquad I_{xy} = -\sum m_j x_j y_j \qquad I_{xz} = -\sum m_j x_j z_j$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

Angular momentum of a rigid body



$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

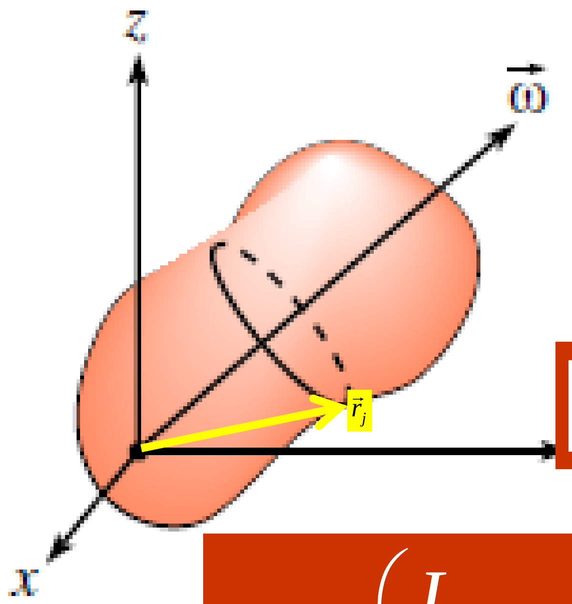
Different from what you learned!

Matrix Equation

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$[L] = [I][\omega]$$

Angular momentum of a rigid body



$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$[L] = [I][\omega]$ **Equivalent to** $\vec{p} = m\vec{v}$ **?**

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Moment of Inertia Matrix

Moment of Inertia Matrix

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2 \right)$$

For a continuous medium,

$$I_{xx} = \int \left(y^2 + z^2 \right) dm$$

$$[I] = \begin{pmatrix} \int \left(y^2 + z^2 \right) dm & -\int xydm & -\int zxdm \\ -\int xydm & \int \left(z^2 + x^2 \right) dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int \left(x^2 + y^2 \right) dm \end{pmatrix}$$

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$\vec{\tau} = [I][\vec{\alpha}]$$

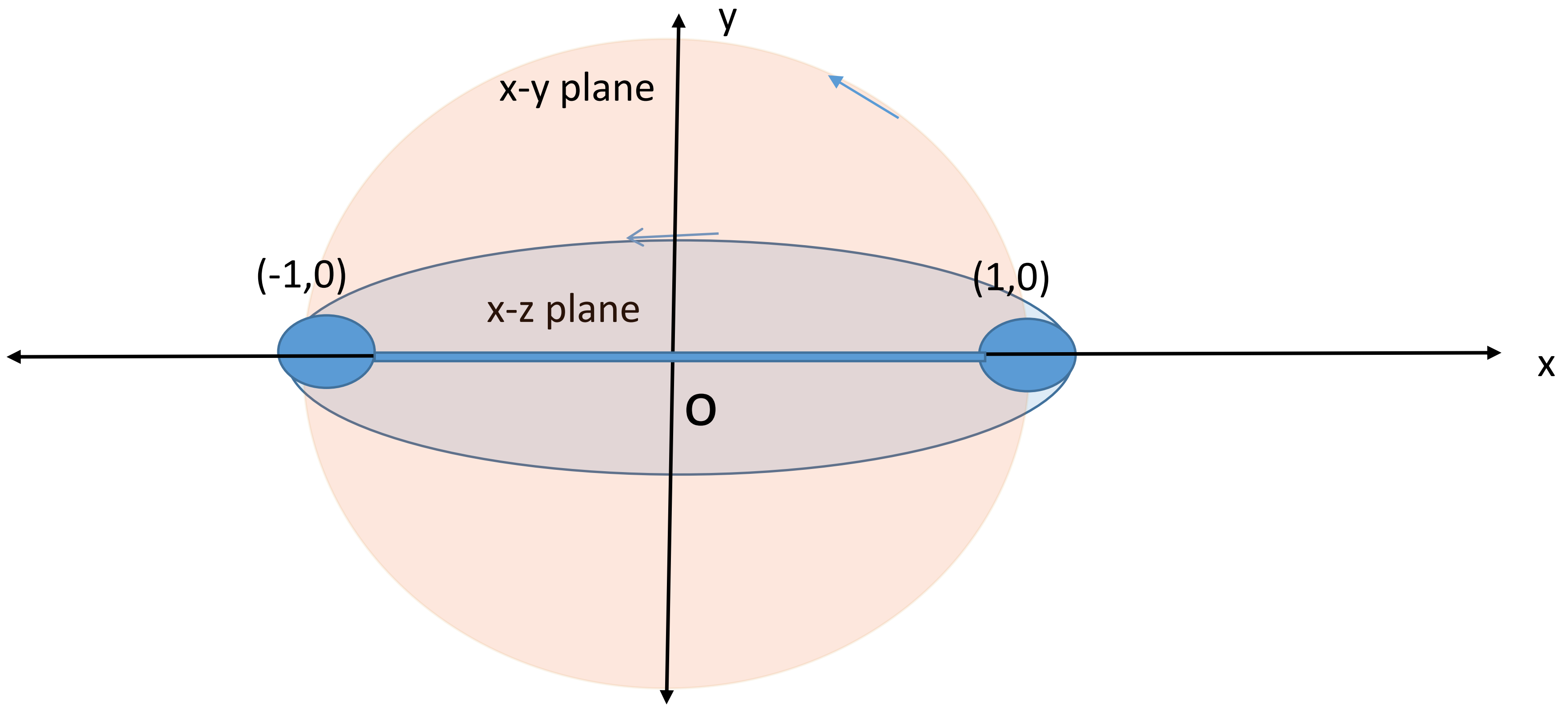
The fact that *I* is a matrix means that $\vec{L}$ and $\vec{\omega}$ do not necessarily point in the same direction.

$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$\tau_z = I_{zx}\alpha_x + I_{zy}\alpha_y + I_{zz}\alpha_z$$

Torque and angular acceleration may not be in the same direction!!!.

Problem 1



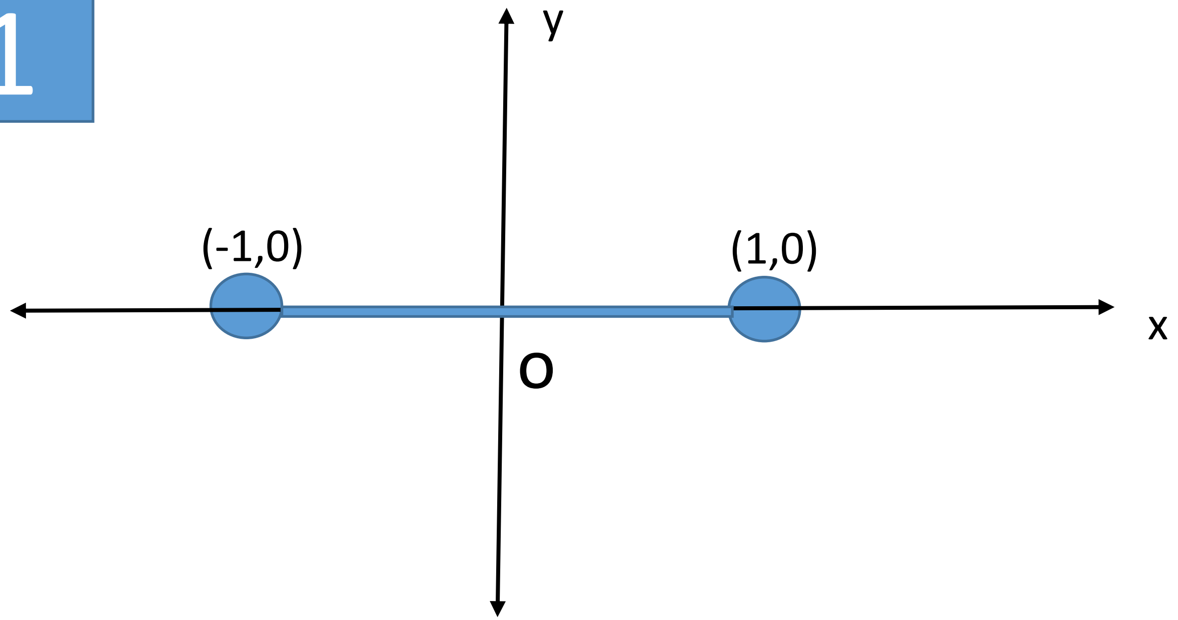
Consider no mass for the interconnecting rod between the two balls. Also consider these balls as point particles.

a. Find the Moment of Inertia Matrix

b. Angular momentum and Torque matrix

c. Physically interpret the same

Example 1



$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2\right) = m(0^2+0^2) + m(0^2+0^2) = 0$$

$$I_{yy} = \sum m_j \left(x_j^2 + z_j^2\right) = m(1^2+0^2) + m(-1^2+0^2) = 2m$$

$$I_{zz} = \sum m_j \left(x_j^2 + y_j^2\right) = m(1^2+0^2) + m(-1^2+0^2) = 2m$$

$$I_{xy} = -\sum m_j x_j y_j = -m\times1\times0 - m\times(-1)\times0 = 0$$

$$I_{yz} = -\sum m_j y_j z_j = -m\times0\times0 - m\times0\times0 = 0$$

$$I_{xz} = -\sum m_j x_j z_j = -m\times1\times0 - m\times(-1)\times0 = 0$$

$$[I] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix}$$

Angular momentum and Torque

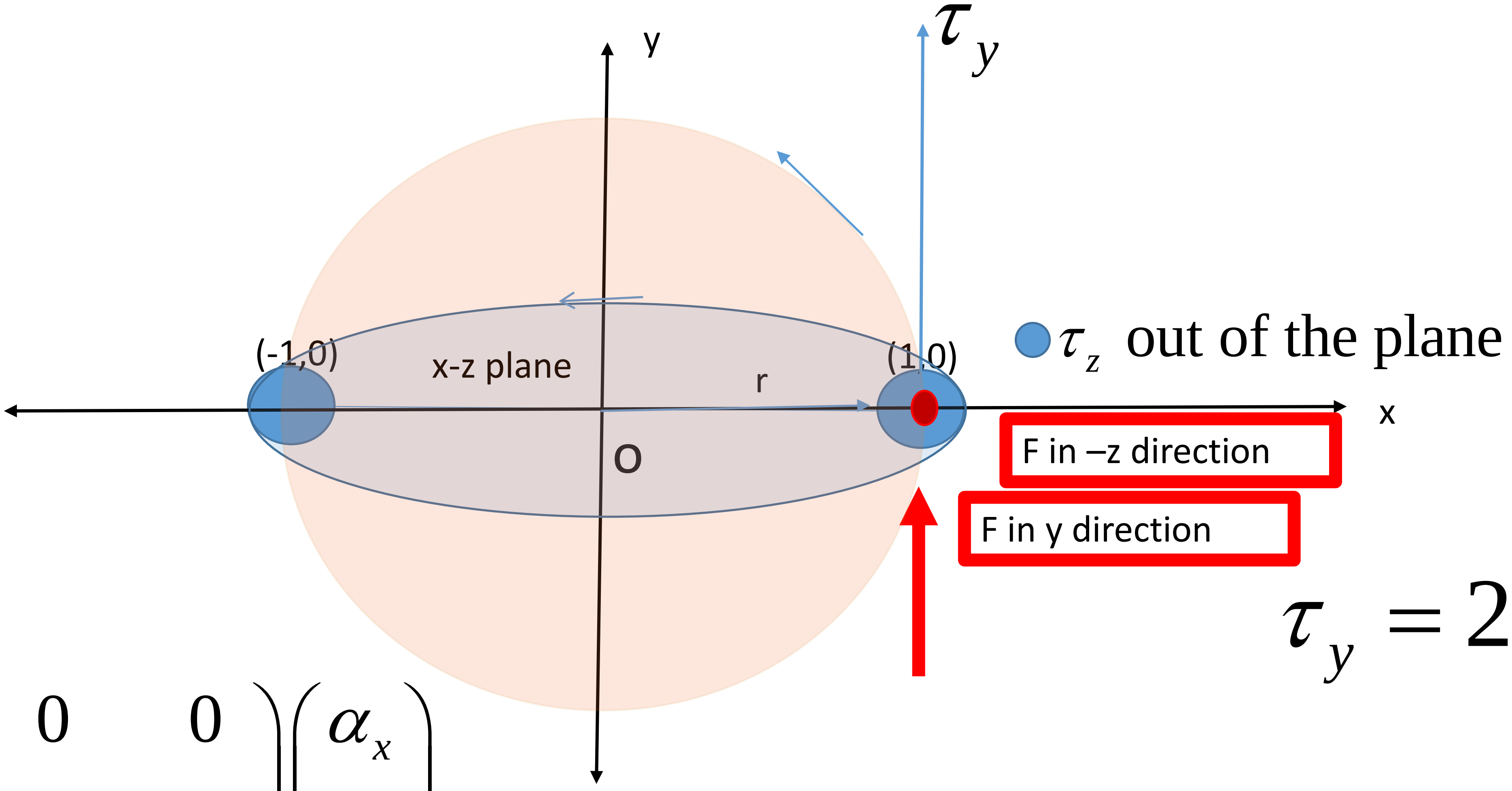
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$\tau_x = 0$$

$$\tau_y = 2m\alpha_y$$

$$\tau_z = 2m\alpha_z$$



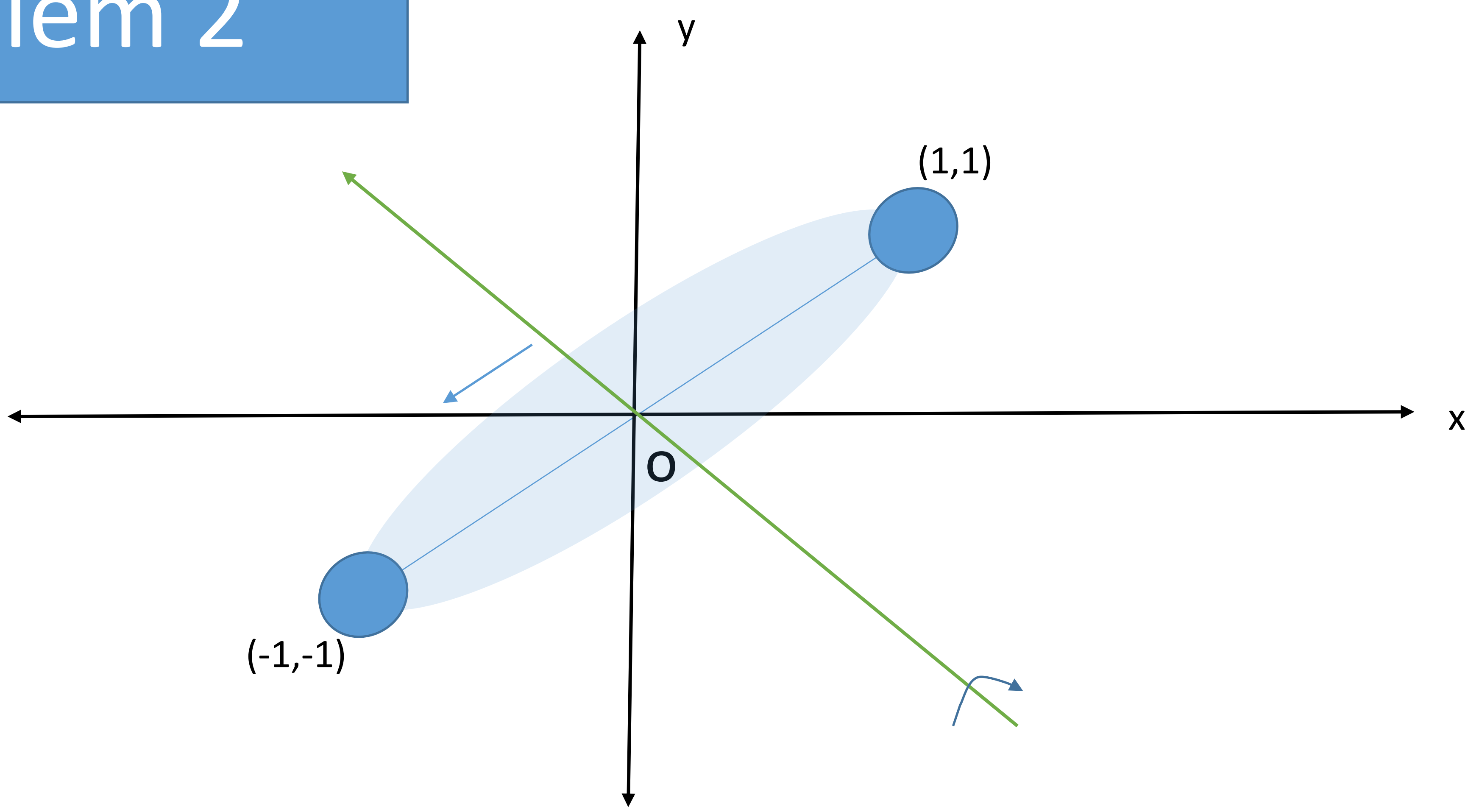
$$\tau_y = 2m\alpha_y$$

$$\tau_z = 2m\alpha_z$$

$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

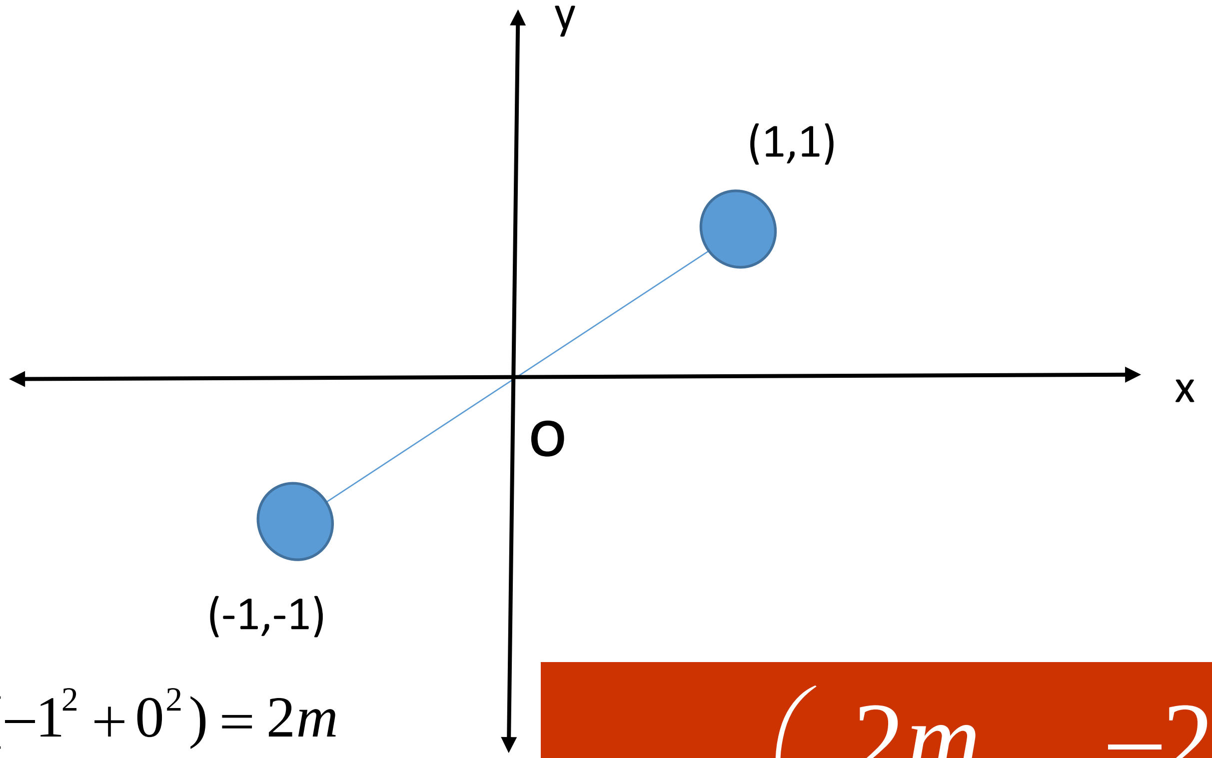
since rod is fixed in x-axis one cannot rotate with respect x-axis.

Problem 2



Consider no mass for the interconnecting rod between the two balls. Also consider these balls as point particles.

a. **Find the Moment of Inertia Matrix**
b. **Angular momentum and torque matrix**
c. **Physically interpret the same for the current motion and if the system is rotated along z-axis**
d. **Find the net angular acceleration in this case also compare it, when the whole system is rotated in x-z plane**



$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2 \right) = m(1^2 + 0^2) + m(-1^2 + 0^2) = 2m$$

$$I_{yy} = \sum m_j \left(x_j^2 + z_j^2 \right) = m(1^2 + 0^2) + m(-1^2 + 0^2) = 2m$$

$$I_{zz} = \sum m_j \left(x_j^2 + y_j^2 \right) = m(1^2 + 1^2) + m(-1^2 + 1^2) = 4m$$

$$I_{xy} = -\sum m_j x_j y_j = -m \times 1 \times 1 - m \times (-1) \times -1 = -2m$$

$$I_{yz} = -\sum m_j y_j z_j = -m \times 1 \times 0 - m \times -1 \times 0 = 0$$

$$I_{xz} = -\sum m_j x_j z_j = -m \times 1 \times 0 - m \times (-1) \times 0 = 0$$

$$[I] = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

Angular momentum and Torque

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

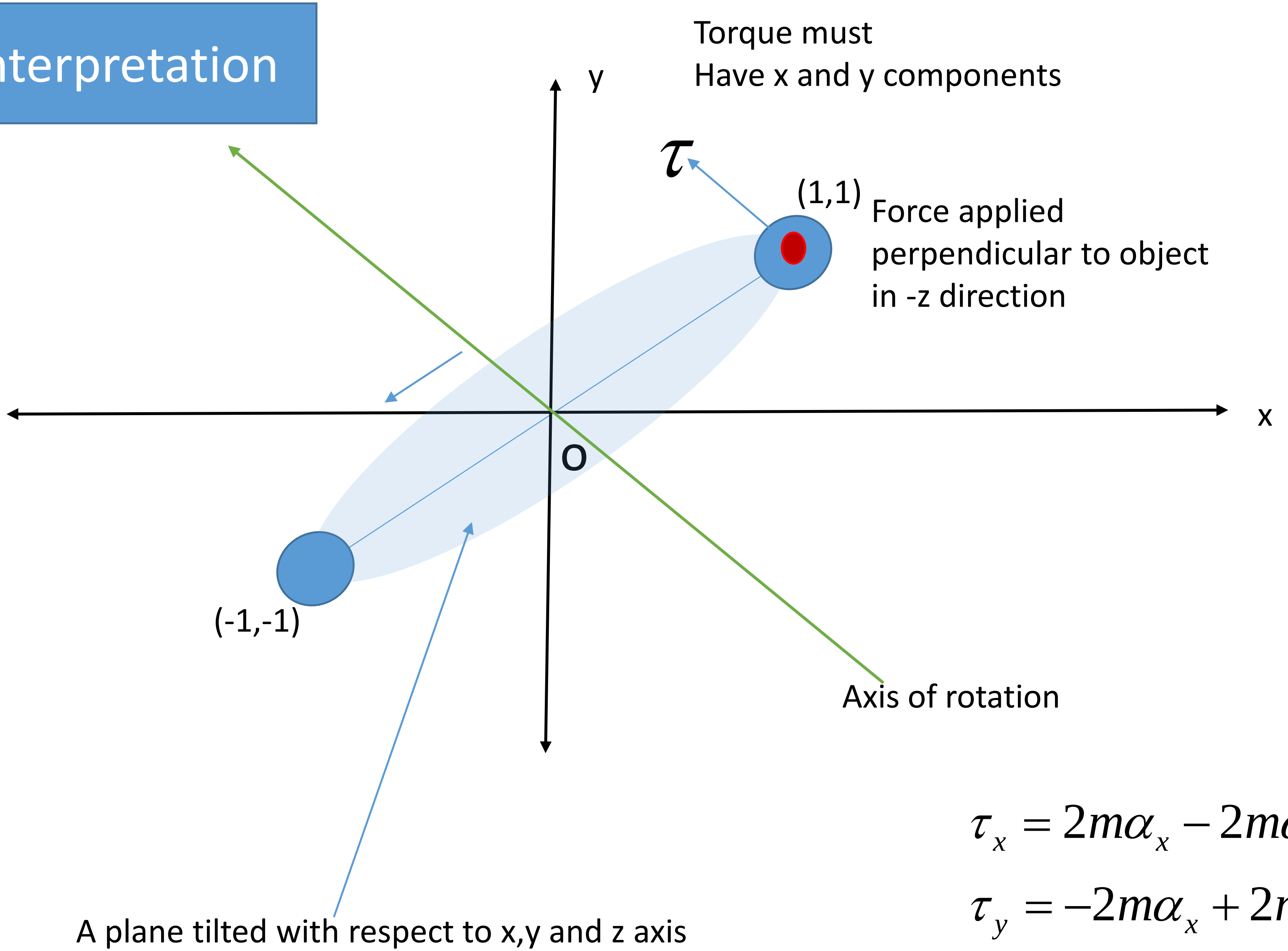
$$\begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$$

$$\tau_x = 2m\alpha_x - 2m\alpha_y$$

$$\tau_y = -2m\alpha_x + 2m\alpha_y$$

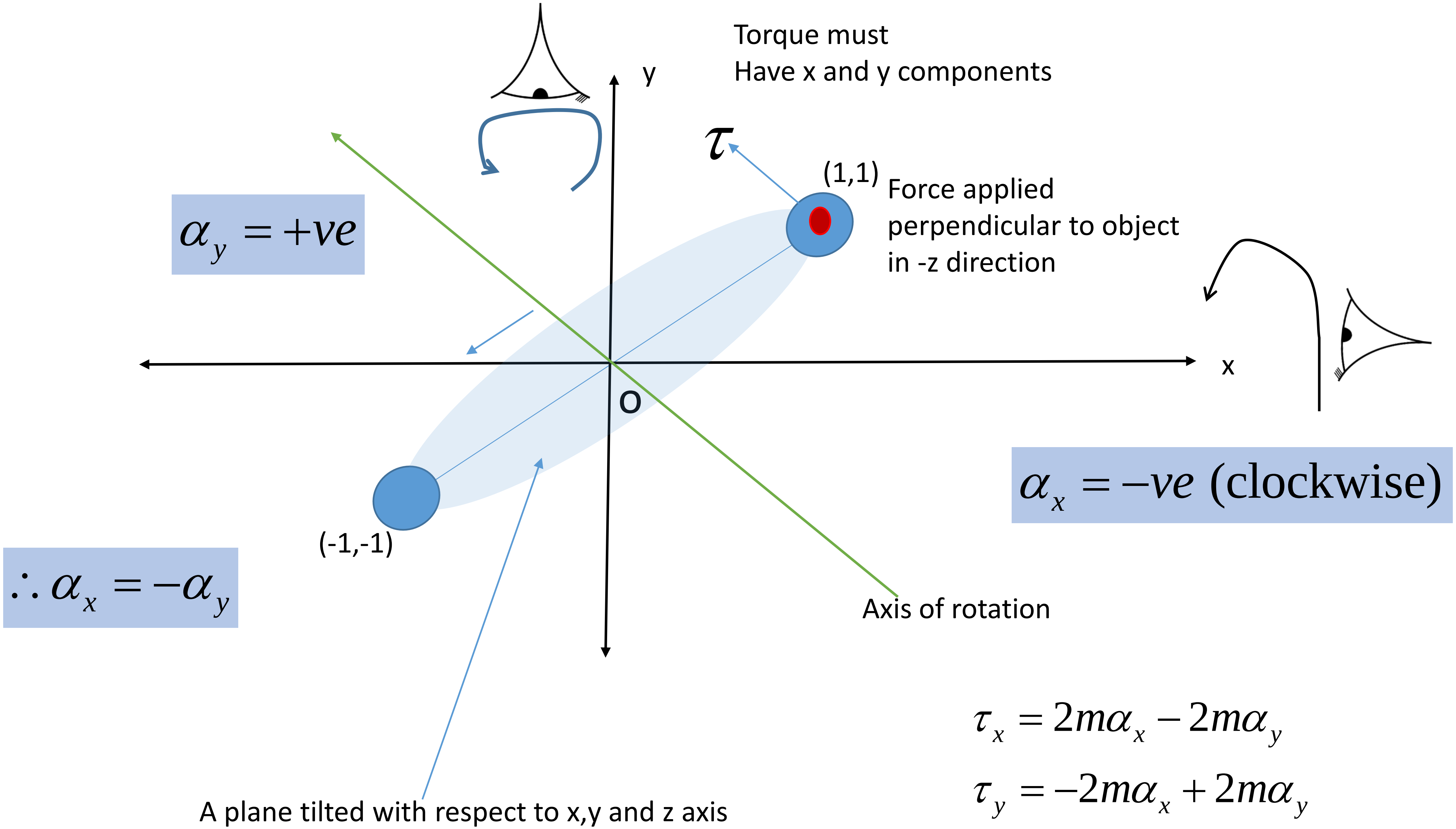
$$\tau_z = 4m\alpha_z$$

Physical Interpretation



$$\tau_x = 2m\alpha_x - 2m\alpha_y$$

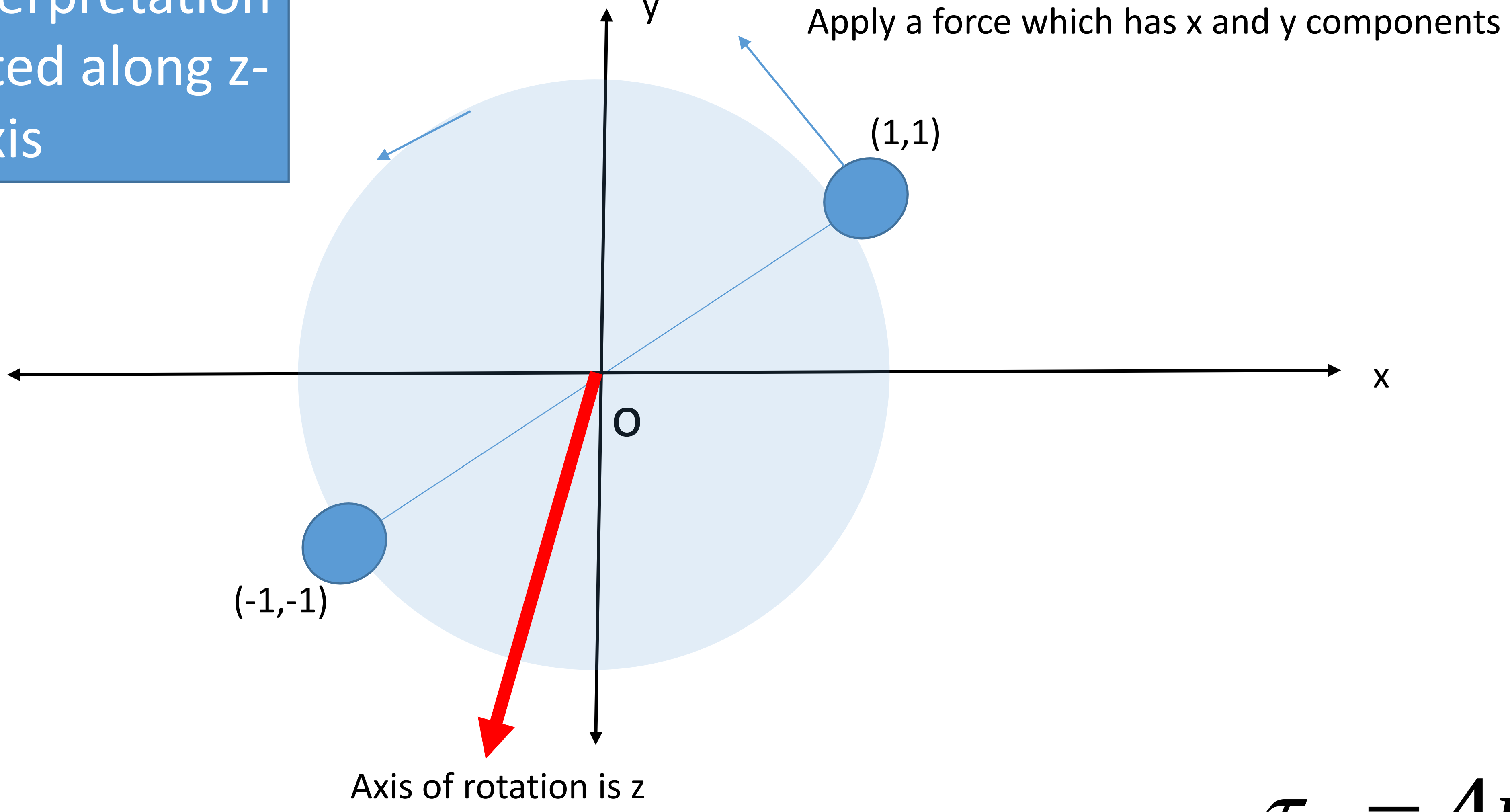
$$\tau_y = -2m\alpha_x + 2m\alpha_y$$



$$\tau_x = 2m\alpha_x - 2m\alpha_y$$

$$\tau_y = -2m\alpha_x + 2m\alpha_y$$

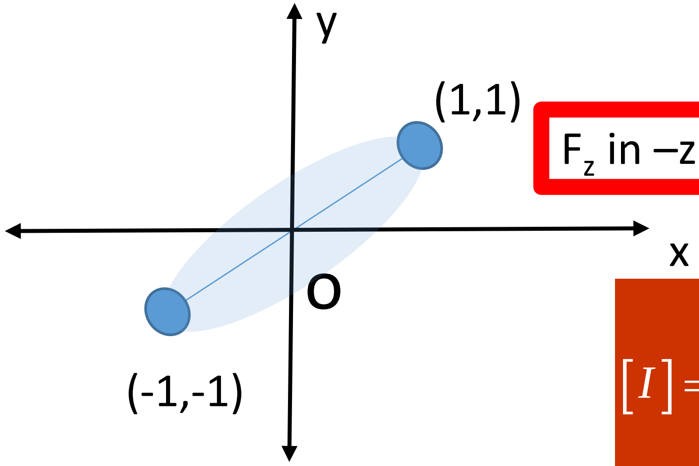
Physical Interpretation When rotated along z-axis



$$\tau_z = 4m\alpha_z$$

Find the net angular acceleration in this case also compare it, when the whole system is rotated in x-z plane

Angular acceleration



F_z in –z direction

$$[I]=\begin{pmatrix} 2m & -2m & 0 \\ -2m & 2m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

$$\tau_x = 2m\alpha_x - 2m\alpha_y$$
$$\tau_y = -2m\alpha_x + 2m\alpha_y$$

$$\vec{\tau} = \vec{r}\times\vec{F} = \begin{vmatrix} i & j & k \\ x & y & 0 \\ 0 & 0 & F_z \end{vmatrix} = F_z y i - F_z x k = \tau_x i + \tau_y j$$

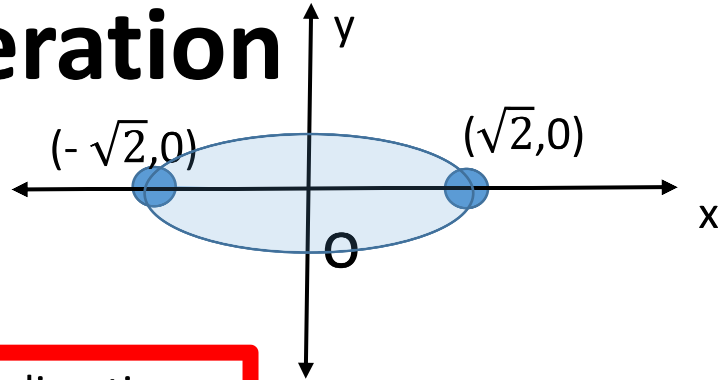
$$\alpha_x = -\alpha_y$$

$$\tau_x = F_z y = 2m\alpha_x - 2m\alpha_y$$
$$\tau_y = -F_z x = -2m\alpha_x + 2m\alpha_y$$

$$\alpha_y = -\frac{F_z y}{4m} \qquad \alpha_x = \frac{F_z y}{4m}$$

y=1

$$|\alpha| = \frac{F_z}{2\sqrt{2}m}$$



F_z in –z direction

$$[I]=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & 4m \end{pmatrix}$$

$$\tau_y = 4m\alpha_y$$

$$xF_z = 4m\alpha_y$$

$$x = \sqrt{2}$$

$$\alpha_y = \frac{xF_z}{4m} \qquad |\alpha| = \frac{F_z}{2\sqrt{2}m}$$