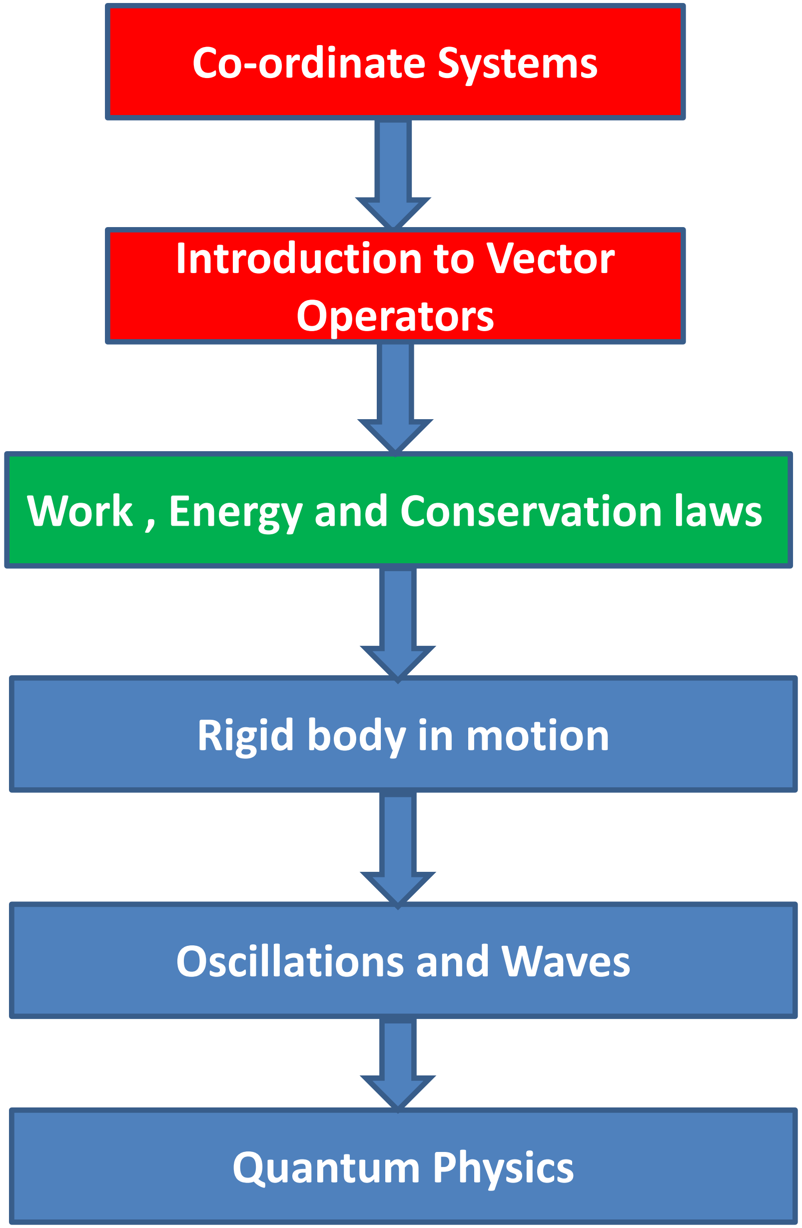
Highlights of the course



Chapter-3:Work , Energy and Conservation laws

Concept of Potential Energy

For a conservative force

$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \bullet \overrightarrow{dr} = function(r_b) - function(r_a)$$

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \bullet \overrightarrow{dr} = -\left[V(r_b) - V(r_a)\right]$$

V(r) is known as potential energy function

$$W_{ba} = K_b - K_a$$

$$K_a + V_a = K_b + V_b$$

This proves that if Force is conservative Total Energy E of the system is independent of Position of particle.

Position '*a*' and '*b*' are arbitrary, hence the above relation is true at any point.

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\left(\vec{r}\,'\right) \bullet \overrightarrow{dr\,'}$$

Necessary conditions for Conservative Force

Curl of F is zero?

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\left(\vec{r}\,'\right) \bullet \overrightarrow{dr}\,'$$

If V(r) is path independent,

$$dV(\vec{r}) = -\vec{F}\left(\vec{r}\right) \bullet \overrightarrow{dr}$$

$$dV(\vec{r}) = -\left[F_x dx + F_y dy + F_z dz \right]$$

Alternatively,

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = -\left[F_x dx + F_y dy + F_z dz \right]$$

$$\nabla V(\vec{r}) \bullet \overrightarrow{dr} = -\vec{F}\left(\vec{r}\right) \bullet \overrightarrow{dr}$$

$$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$$

Curl of F is $\vec{\nabla} \times \vec{F}\left(\vec{r}\right) = -\left[\vec{\nabla} \times \nabla V(\vec{r}) \right]$

Necessary conditions for Conservative Force

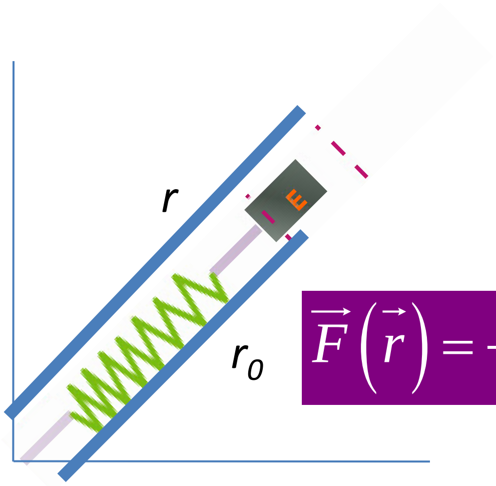
Curl of F is zero?

Curl of F is $\vec{\nabla}\times\vec{F}\left(\vec{r}\right)=-\left[\vec{\nabla}\times\nabla V(\vec{r})\right]$

$$-\left[\vec{\nabla}\times\nabla V(\vec{r})\right]=-\begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z}\end{vmatrix}$$

Examples

- Find potential energy of Spring Force



$$\vec{F}\left(\vec{r}\right) = -k(r - r_0)\hat{e}_r$$

Since the central force is conservative,

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} -k(r - r_0)dr$$

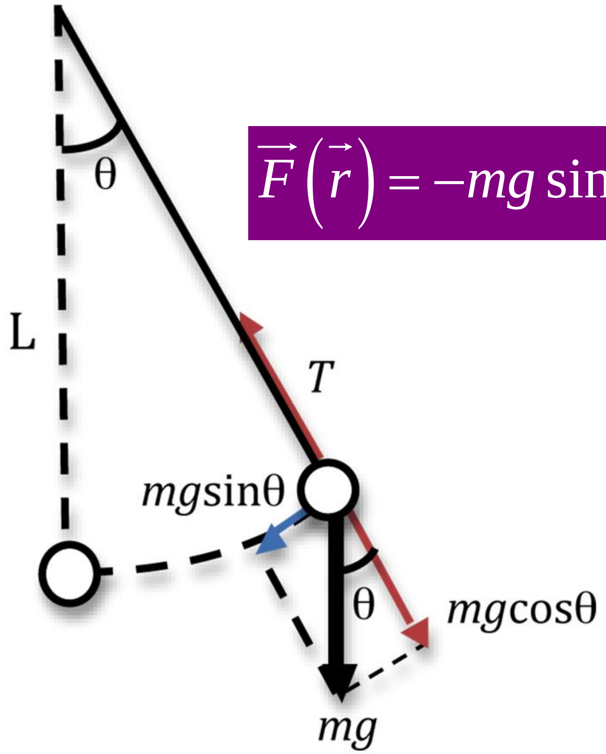
$$V(\vec{r}) - V(\vec{r}_0) = \frac{1}{2}k(r - r_0)^2$$

Choosing potential energy to be zero at equilibrium,,

$$V(r) = \frac{1}{2}k(r - r_0)^2$$

Examples

- Find potential energy of Simple pendulum



$$\vec{F}\left(\vec{r}\right) = -mg\sin\theta\hat{e}_\theta$$

$$V(\vec{r}) - V(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} -mg\sin\theta\hat{e}_\theta.\left[dr\hat{e}_r + rd\theta\hat{e}_\theta\right]$$

$$V(\vec{r}) - V(\vec{r_0}) = \int_0^\theta mgl\sin\theta d\theta$$

$$V(\vec{r}) - V(\vec{r_0}) = mgl(1-\cos\theta)$$

Choosing potential energy to be zero at equilibrium,,

$$V(\vec{r}) = mgl(1-\cos\theta)$$

Examples

- Central force is conservative by showing that curl F=0

$$\overrightarrow{F} = f(r)\hat{e}_r$$

$$\overrightarrow{\nabla}\times\overrightarrow{F} = \frac{\widehat{e}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial\phi}\right) +$$

$$\frac{\widehat{e}_\theta}{r\sin\theta}\left(\frac{\partial F_r}{\partial\phi} - \sin\theta\frac{\partial\left(rF_\phi\right)}{\partial r}\right) +$$

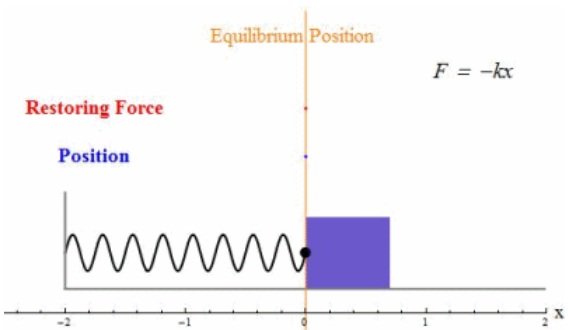
$$\frac{\widehat{e}_\phi}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)$$

Concept of equilibrium

$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$ is useful for visualizing the stability of a system

Examples

1-D Harmonic Oscillator

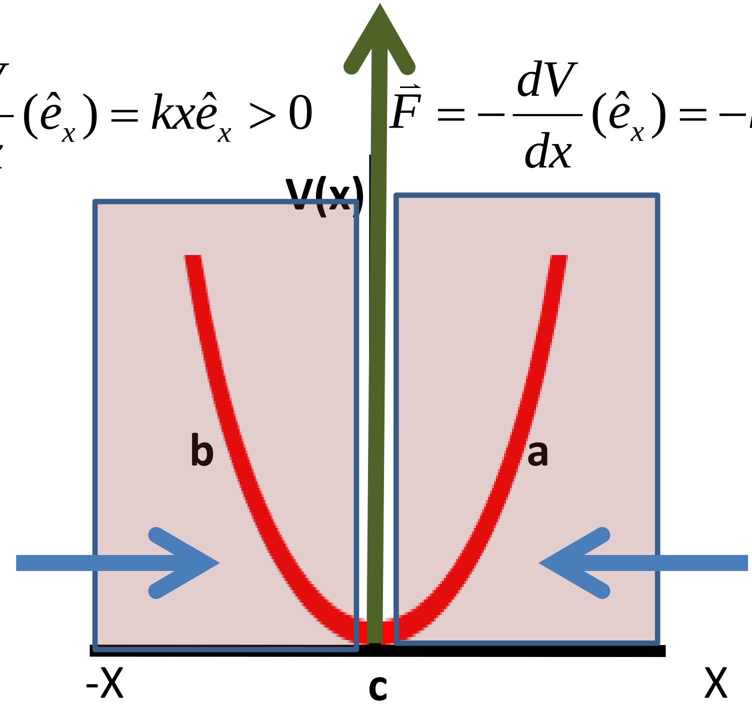


$$V(x) = \frac{1}{2}kx^2$$

$$\vec{F} = -\frac{dV}{dx}(\hat{e}_x) = 0$$

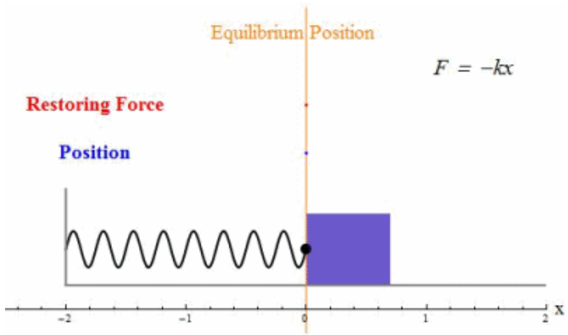
$$\vec{F} = -\frac{dV}{dx}(\hat{e}_x) = kx\hat{e}_x > 0$$

$$\vec{F} = -\frac{dV}{dx}(\hat{e}_x) = -kx\hat{e}_x < 0$$



$\frac{dV}{dx} = 0$ → Equilibrium of the system

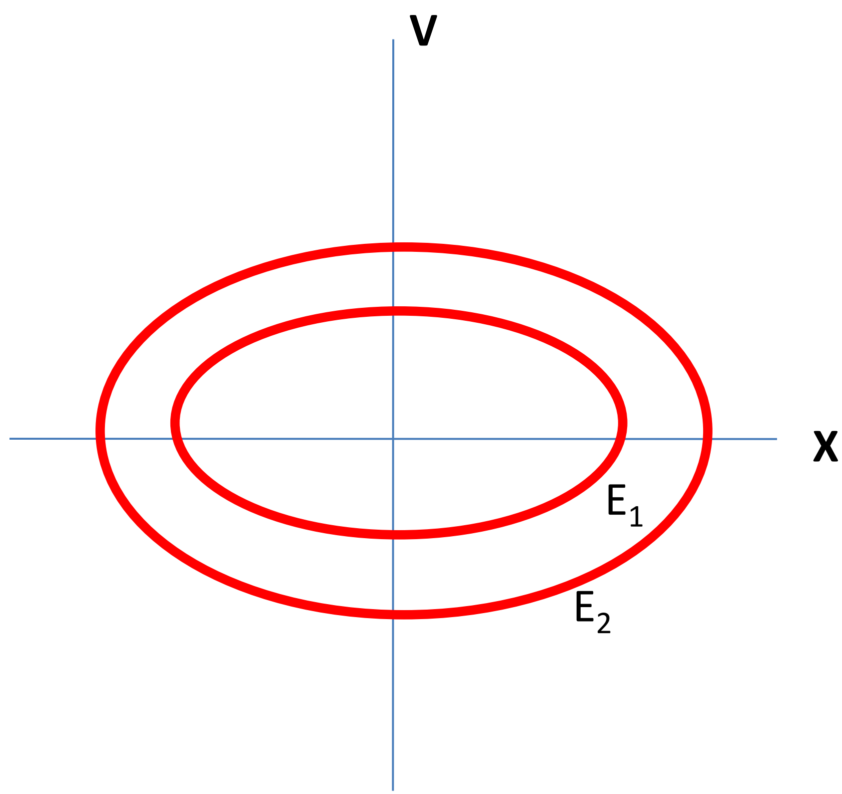
Velocity Vs. Position plot



$$E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

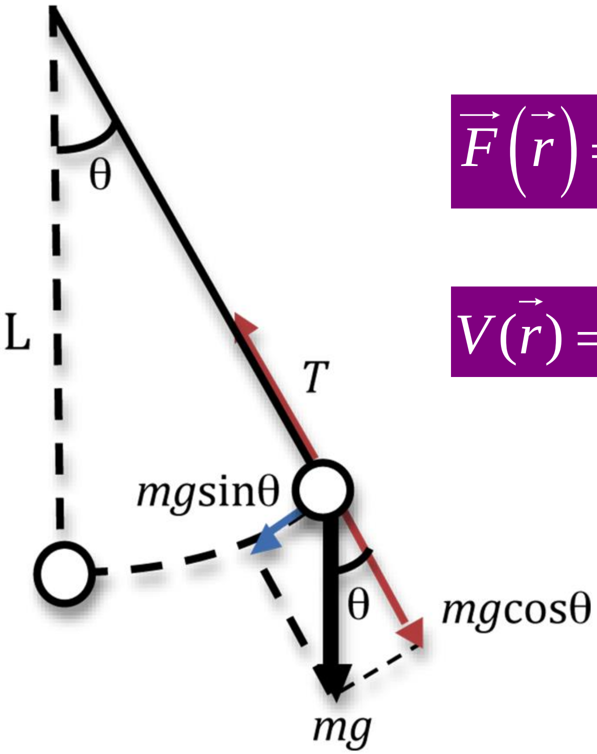
Plot for energy = E_1

$E_2 > E_1$



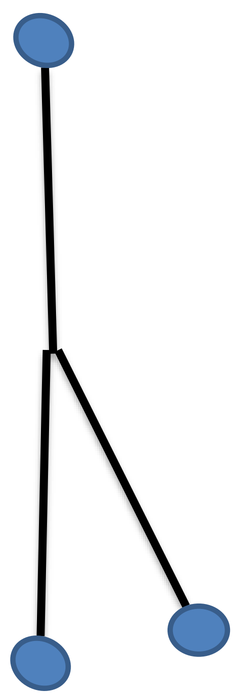
Concept of equilibrium

Simple Pendulum

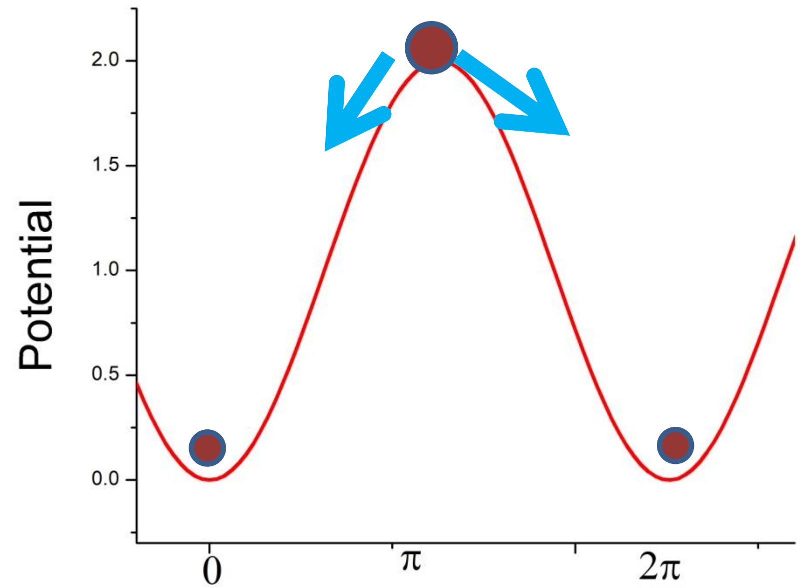


$$\vec{F}\left(\vec{r}\right) = -mg\sin\theta\hat{e}_{\theta}$$

$$V(\vec{r}) = mgl(1-\cos\theta)$$



Unstable equilibrium



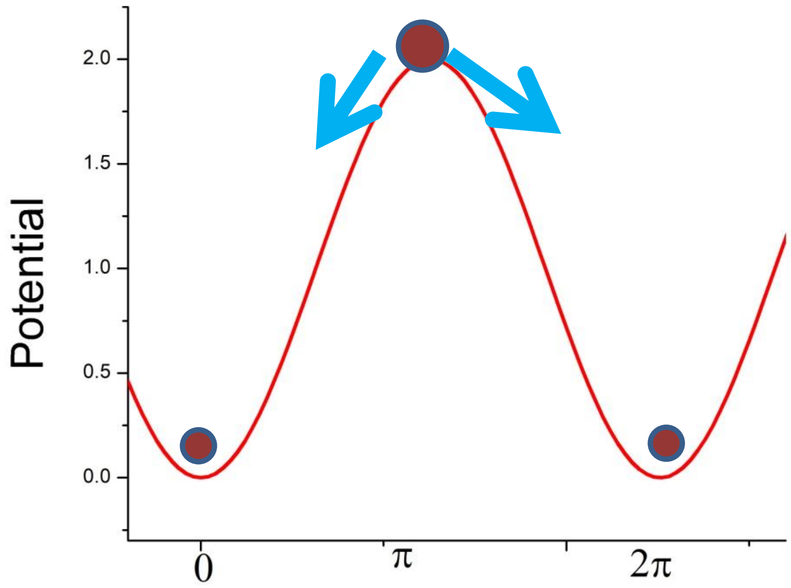
$\theta = \pi$

$\theta = 0$

$$V(\vec{r}) = mgl(1-\cos\theta)$$

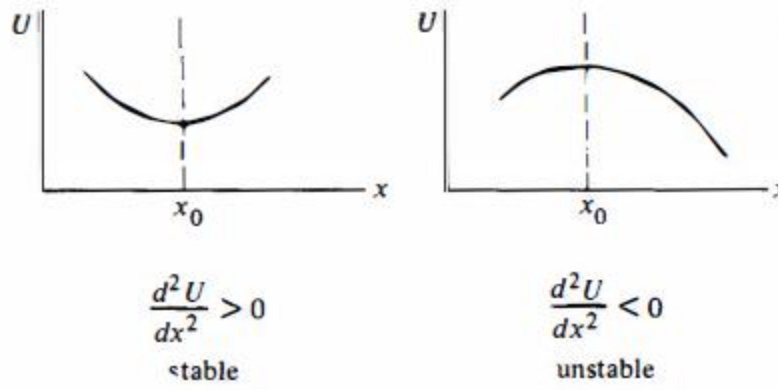
$$\frac{dV(\vec{r})}{d\theta} = mgl\sin\theta$$

$$\frac{d^2V(\vec{r})}{d\theta^2} = mgl\cos\theta$$



$$\frac{d^2V(\vec{r})}{d\theta^2} = mgl(\theta = 0)$$

$$\frac{d^2V(\vec{r})}{d\theta^2} = -mgl(\theta = \pi)$$



Plot $\dot{\theta}$ v/s θ ?

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - (1 - \frac{\theta^2}{2!} + ...)$$

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl \frac{\theta^2}{2!}$$

$$\frac{\dot{\theta}^2}{2E / ml^2} + \frac{\theta^2}{2E / mgl} = 1$$

Equation of ellipse

For small values of ɵ the equation is of ellipse. But as ɵ becomes larger it Deviates from an ellipse

$$E = \frac{1}{2} ml^2 \dot{\theta}^2 + mgl \frac{\theta^2}{2!} - mgl \frac{\theta^4}{4!} + ...$$

$$\frac{\dot{\theta}^2}{2E / ml^2} + \frac{\theta^2}{2E / mgl} - \frac{\theta^4}{24E / mgl} + ... = 1$$

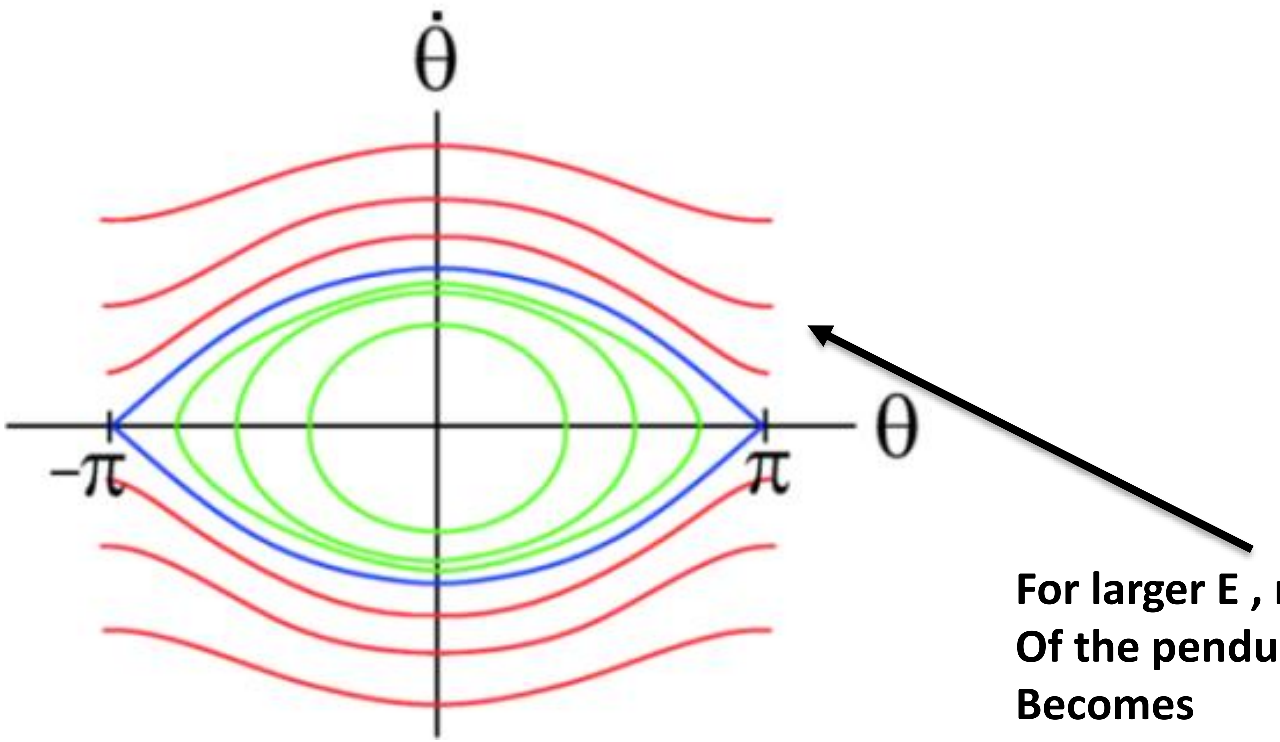
Plot $\dot{\theta}$ v/s θ ?

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \frac{\theta^2}{2!}$$

$$\frac{\dot{\theta}^2}{2E/ml^2} + \frac{\theta^2}{2E/mgl} = 1$$

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \frac{\theta^2}{2!} - mgl \frac{\theta^4}{4!} + \ldots\ldots$$

$$\frac{\dot{\theta}^2}{2E/ml^2} + \frac{\theta^2}{2E/mgl} - \frac{\theta^4}{24E/mgl} + \ldots = 1$$



For larger E , motion Of the pendulum Becomes Circular motion