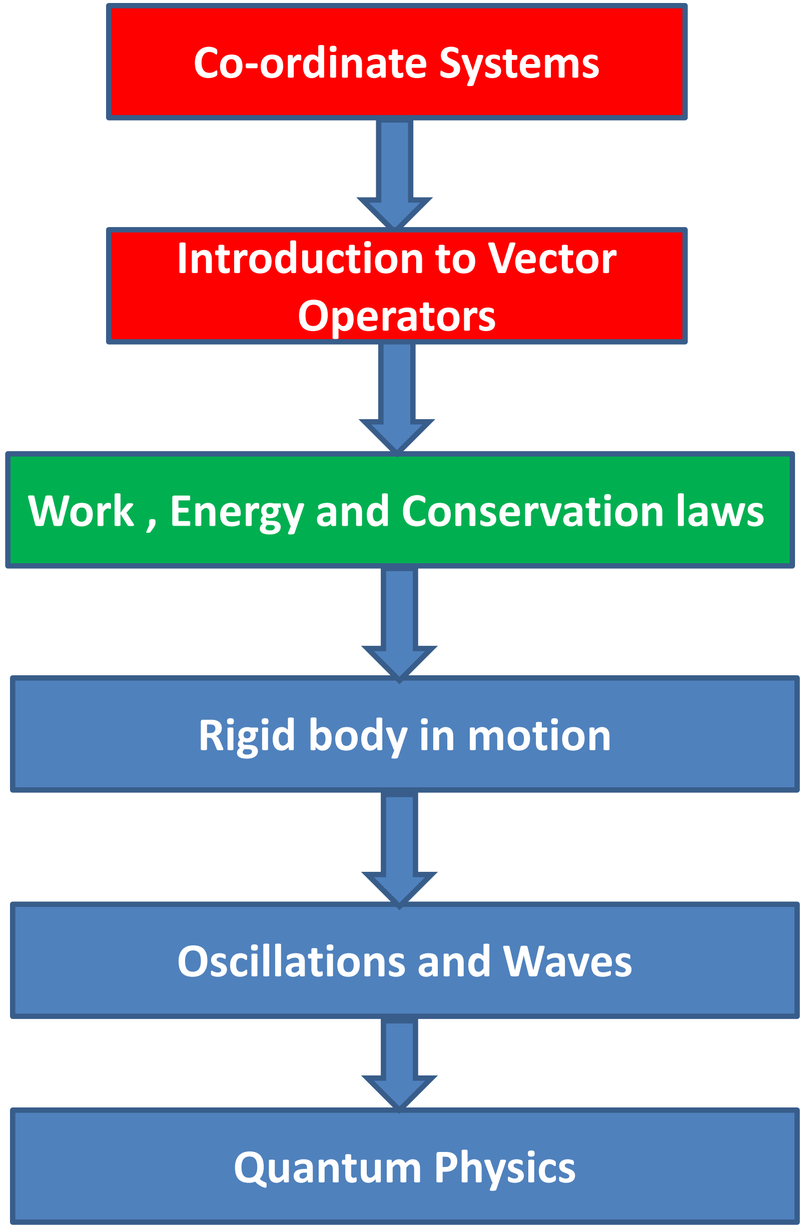
Highlights of the course



Chapter-3:Work , Energy and Conservation laws

Predicting the motion of a system under known constraints

$$m\frac{d^2x}{dt^2} = F(x)$$

Solve differential equation

Work, energy and conserved quantities

Work Energy Theorem from Force Equation

$$m\int_{x_a}^{x_b}\frac{dv}{dt}dx=\int_{x_a}^{x_b}F(x)dx$$

Change variable from *x* to *t*

$$m\int_{t_a}^{t_b}\frac{dv}{dt}vdt=\int_{x_a}^{x_b}F(x)dx$$

$$m\int_{t_a}^{t_b}\frac{d}{dt}\left(\frac{1}{2}v^2\right)dt=\int_{x_a}^{x_b}F(x)dx$$

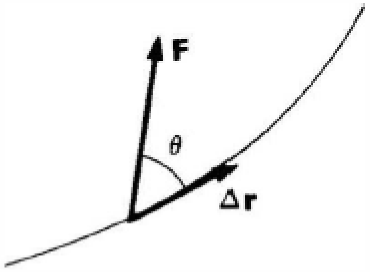
Work Energy Theorem from Force Equation

$$W_{ba} = K_b - K_a$$

The change in kinetic energy equals work done on the particle

Work Energy Theorem in 3D

$$\overrightarrow{F} = m\frac{d\vec{v}}{dt}$$

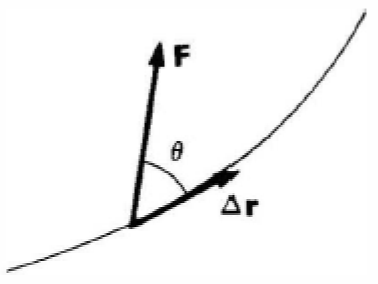


$$\int_a^b \overrightarrow{F} \bullet \overrightarrow{dr} = \int_a^b m\frac{d\vec{v}}{dt} \bullet \overrightarrow{dr}$$

$$W_{ba} = K_b - K_a$$

W_{ba} : Work done on the particle by the total force

Work Energy Theorem- Physical interpretation



$$\int_a^b \vec{F} \bullet \vec{dr} = \frac{m}{2}\left[v_b^{\ 2} - v_a^{\ 2}\right]$$

$$W_{ba} = K_b - K_a$$

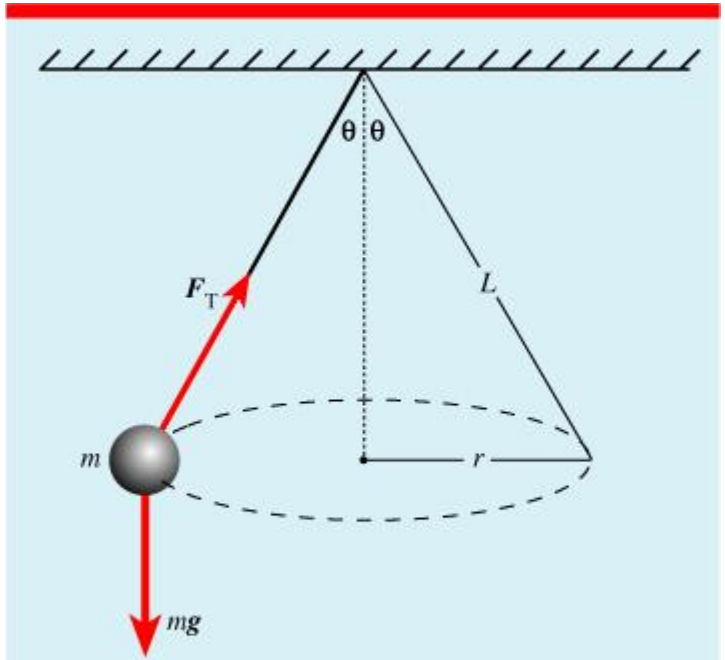
For infinitesimally small displacement $\vec{\Delta r}$,

$$\Delta W = \vec{F} \bullet \vec{\Delta r}$$

$F_\perp$ does no work!!

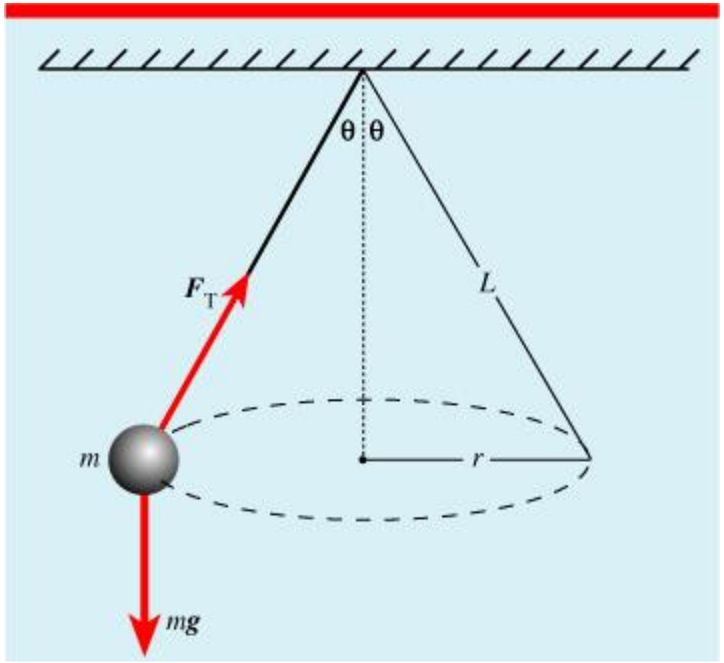
For a finite displacement, the work done on the particle is the sum of the contributions $\Delta W = F_{\parallel}\Delta r$ from each segment of the path, in the limit where the size of each segment approaches zero.

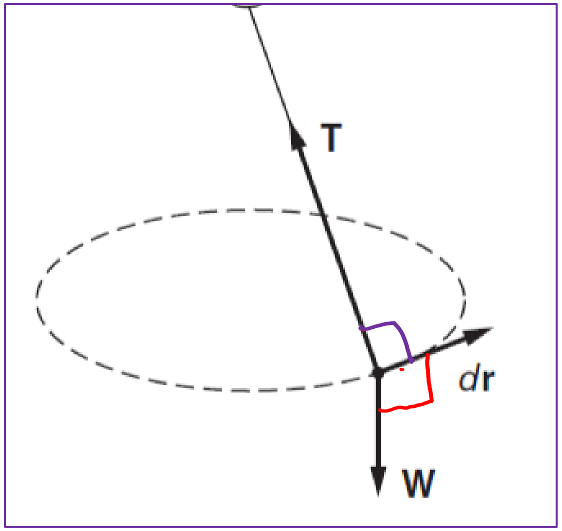
Conical Pendulum



Mass M is fixed to the end of a rod of length l and negligible mass that is pivoted to swing. The mass moves with constant speed in a circular path of constant radius. Find the work done by the string force and weight force.

Conical Pendulum





$$W_{ba} = K_b - K_a$$

Since the kinetic energy is constant, the work energy theorem tells that no net work is being done on the mass.

String force and weight force separately do no work.

Work Energy Theorem- Physical interpretation

$$W_{ba} = \int_a^b \vec{F} \bullet \vec{dr}$$

Work Energy Theorem- is a mathematical consequence of Newton's second law.

Evaluation of this integral depends on knowing what path the particle actually follows

It needs to know the solution in order to apply the theorem.

Solution → Work-Energy Theorem

Constrained motion

Conservative forces

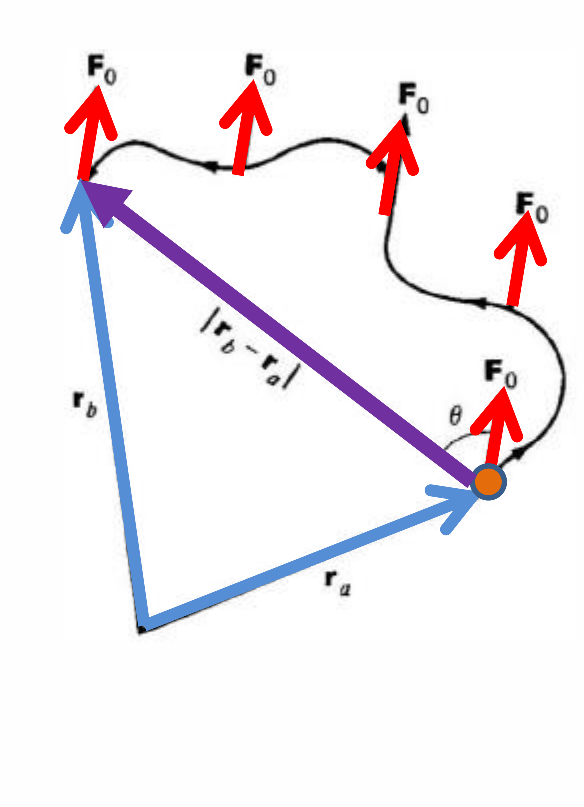
Conservative Forces

Definition:

The forces whose work integral does not depend on the particular path but only on the end points are called as Conservative Forces [Implications will be shown shortly]

Examples are work done by a uniform force, central force etc

Work done by a uniform Force



$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \overrightarrow{F} \bullet \overrightarrow{dr}$$

$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} F_0 \hat{n} \bullet \overrightarrow{dr}$$

$$W_{ba} = F_0 \hat{n} \bullet \int_{\vec{r}_a}^{\vec{r}_b} \overrightarrow{dr}$$

$$W_{ba} = F_0 \hat{n} \bullet \left(\vec{r}_b - \vec{r}_a \right)$$

$$W_{ba} = F_0 \cos\theta \left| \vec{r}_b - \vec{r}_a \right|$$

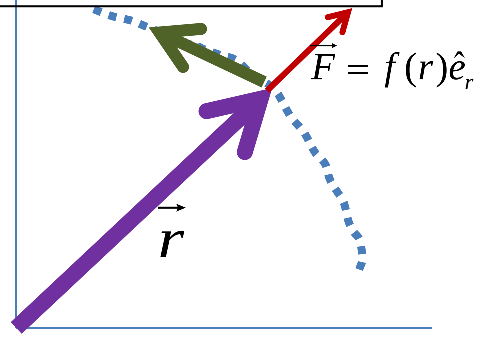
For a constant force, work done only depends on the net displacement and not on the path followed.

Work Done by a Central Force

Central force is a radial force which depends only on the distance from the origin

Motion in a plane

$$\overrightarrow{dr} = dr\hat{e}_r + rd\theta\hat{e}_\theta$$



$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \bullet \overrightarrow{dr}$$

$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} f(r)\hat{e}_r \bullet \left[dr\hat{e}_r + rd\theta\hat{e}_\theta\right]$$

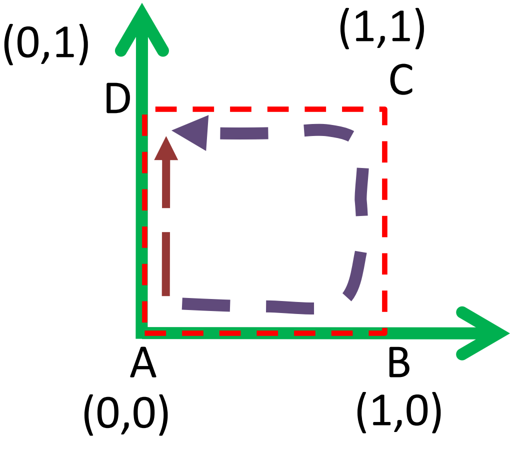
$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} f(r)dr$$

It follows that the work by a central force around a closed path is zero.

Note that work done only depends on initial and final radial distances, and not on the particular path.

Non- Conservative Force

- **The forces whose work is different for different paths between the initial and final points.**
- **Example is Friction force**
  - **Different paths will offer different friction**



Concept of Potential Energy

For a conservative force

$$W_{ba} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \bullet \vec{dr} = function(r_b) - function(r_a)$$

$$\int_{\vec{r}_a}^{\vec{r}_b} \vec{F} \bullet \vec{dr} = -\left[V(r_b) - V(r_a)\right]$$

V(r) is known as potential energy function

$$W_{ba} = K_b - K_a$$

$$K_a + V_a = K_b + V_b$$

This proves that if Force is conservative Total Energy E of the system is independent of Position of particle.

Position *'a'* and *'b'* are arbitrary, hence the above relation is true at any point.

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\left(\vec{r}\,'\right) \bullet \vec{dr}\,'$$

Necessary conditions for Conservative Force

Curl of F is zero?

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}\left(\vec{r}\,'\right) \bullet \overrightarrow{dr}\,'$$

If V(r) is path independent,

$$dV(\vec{r}) = -\vec{F}\left(\vec{r}\right) \bullet \overrightarrow{dr}$$

$$dV(\vec{r}) = -\left[F_x dx + F_y dy + F_z dz\right]$$

Alternatively,

$$\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = -\left[F_x dx + F_y dy + F_z dz\right]$$

$$\nabla V(\vec{r}) \bullet \overrightarrow{dr} = -\vec{F}\left(\vec{r}\right) \bullet \overrightarrow{dr}$$

$$\vec{F}\left(\vec{r}\right) = -\nabla V(\vec{r})$$

Curl of F is $\vec{\nabla} \times \vec{F}\left(\vec{r}\right) = -\left[\vec{\nabla} \times \nabla V(\vec{r})\right]$

Necessary conditions for Conservative Force

Curl of F is zero?

Curl of F is $\vec{\nabla}\times\vec{F}\left(\vec{r}\right)=-\left[\vec{\nabla}\times\nabla V(\vec{r})\right]$

$$-\left[\vec{\nabla}\times\nabla V(\vec{r})\right]=-\begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z}\end{vmatrix}$$

Announcement

First Quiz on 16th April (Saturday, 10:AM)

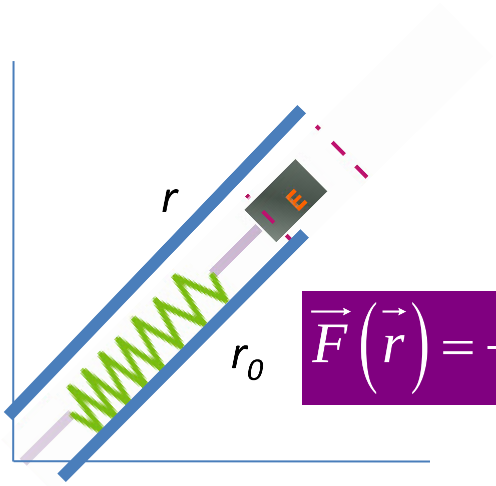
Online Quiz, 50 minutes duration

Maximum marks: 10

Nature: Descriptive

Examples

- Find potential energy of Spring Force



$$\vec{F}\left(\vec{r}\right) = -k(r - r_0)\hat{e}_r$$

Since the central force is conservative,

$$V(\vec{r}) - V(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} -k(r - r_0)dr$$

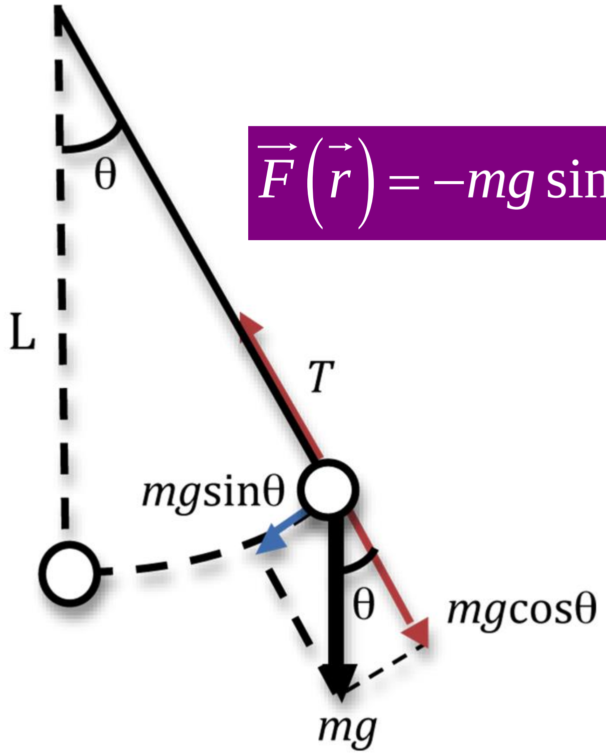
$$V(\vec{r}) - V(\vec{r_0}) = \frac{1}{2}k(r - r_0)^2$$

Choosing potential energy to be zero at equilibrium,,

$$V(r) = \frac{1}{2}k(r - r_0)^2$$

Examples

- Find potential energy of Simple pendulum



$$\vec{F}\left(\vec{r}\right) = -mg\sin\theta\hat{e}_\theta$$

Since the central force is conservative,

$$V(\vec{r}) - V(\vec{r_0}) = -\int_{\vec{r_0}}^{\vec{r}} -mg\sin\theta\hat{e}_\theta.\left[dr\hat{e}_r + rd\theta\hat{e}_\theta\right]$$

$$V(\vec{r}) - V(\vec{r_0}) = \int_0^\theta mgl\sin\theta d\theta$$

$$V(\vec{r}) - V(\vec{r_0}) = mgl(1-\cos\theta)$$

Choosing potential energy to be zero at equilibrium,,

$$V(\vec{r}) = mgl(1-\cos\theta)$$

Examples

- Central force is conservative by showing that curl F=0

$$\overrightarrow{F} = f(r)\hat{e}_r$$

$$\overrightarrow{\nabla} \times \overrightarrow{F} = \frac{\hat{e}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial\phi}\right) +$$

$$\frac{\hat{e}_\theta}{r\sin\theta}\left(\frac{\partial F_r}{\partial\phi} - \sin\theta\frac{\partial\left(rF_\phi\right)}{\partial r}\right) +$$

$$\frac{\hat{e}_\phi}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)$$

Examples

- Central force is conservative by showing that curl F=0

$$\vec{F} = f(r)\hat{e}_r$$

$$\vec{F} = f(r)\left[\frac{x}{r}\hat{e}_x + \frac{y}{r}\hat{e}_y + \frac{z}{r}\hat{e}_z\right]$$

$$\left[\vec{\nabla}\times\vec{F}\right]_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial\left(\frac{yf(r)}{r}\right)}{\partial x} - \frac{\partial\left(\frac{xf(r)}{r}\right)}{\partial y}$$

$$\frac{\partial\left(\frac{yf(r)}{r}\right)}{\partial x} = \frac{y}{r} f'(r)\frac{\partial r}{\partial x} + yf(r)\frac{\partial\left(\frac{1}{r}\right)}{\partial x}$$

$$\left[\vec{\nabla}\times\vec{F}\right]_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial\left(\frac{yf(r)}{r}\right)}{\partial x} - \frac{\partial\left(\frac{xf(r)}{r}\right)}{\partial y} = 0$$

Likewise for the x and y components