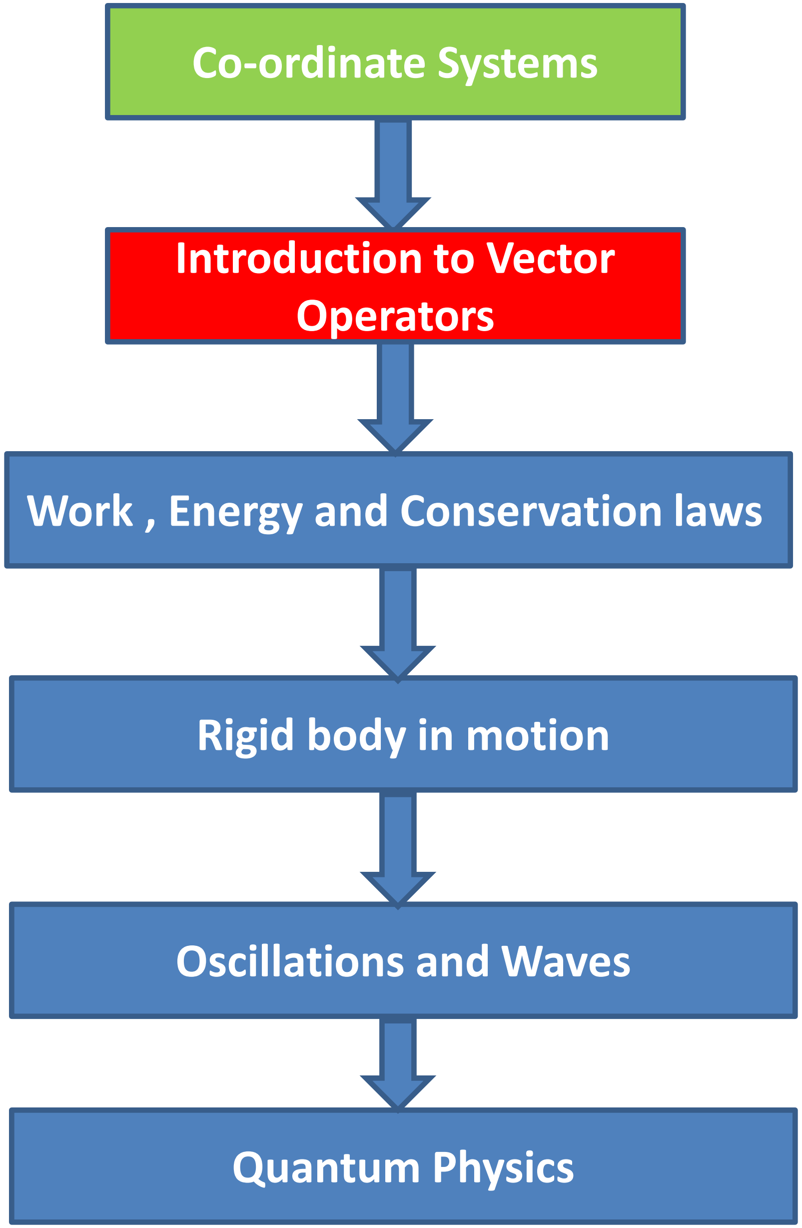
Highlights of the course



Chapter-2: Introduction to Vector Operators

Gradient, Divergence and Curl

Del operator in different coordinate system

Cartesian Coordinate system

$$\vec{\nabla} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Cylindrical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

Divergence in Spherical Polar coordinate system

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$\vec{A} = \hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi$$

$$\vec{\nabla} \bullet \vec{A} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right) . \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right)$$

$$\hat{e}_r \bullet \frac{\partial}{\partial r} \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right) +$$

$$\hat{e}_\theta \bullet \frac{1}{r} \frac{\partial}{\partial \theta} \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right) +$$

$$\hat{e}_\phi \bullet \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right)$$

Divergence in Spherical polar Coordinate system

$$\hat{e}_r \cdot \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\phi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right)$$

$$\hat{e}_r \cdot \left(\hat{e}_r \frac{\partial A_r}{\partial r} + A_r \frac{\partial \hat{e}_r}{\partial r}\right) +$$

Derivatives of Unit Vectors

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\frac{d\hat{e}_r}{dr} = 0 \qquad \frac{d\hat{e}_\theta}{dr} = 0 \qquad \frac{d\hat{e}_\phi}{dr} = 0$$

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta \qquad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r \qquad \frac{d\hat{e}_\phi}{d\theta} = 0$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi \qquad \frac{d\hat{e}_\theta}{d\phi} = \cos\theta\hat{e}_\phi \qquad \frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

Divergence in Spherical polar Coordinate system

$$\hat{e}_r \cdot \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\phi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right)$$

$$\hat{e}_r \cdot \left(\hat{e}_r \frac{\partial A_r}{\partial r} + A_r \frac{\partial \hat{e}_r}{\partial r}\right) +$$

$$\hat{e}_r \cdot \left(\hat{e}_\theta \frac{\partial A_\theta}{\partial r} + A_\theta \frac{\partial \hat{e}_\theta}{\partial r}\right) +$$

$$\hat{e}_r \cdot \left(\hat{e}_\phi \frac{\partial A_\phi}{\partial r} + A_\phi \frac{\partial \hat{e}_\phi}{\partial r}\right)$$

and proceeding further.....

Divergence in Spherical polar Coordinate system

$$\vec{\nabla} \bullet \vec{A} = \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left(A_\theta \sin \theta \right) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \right)$$

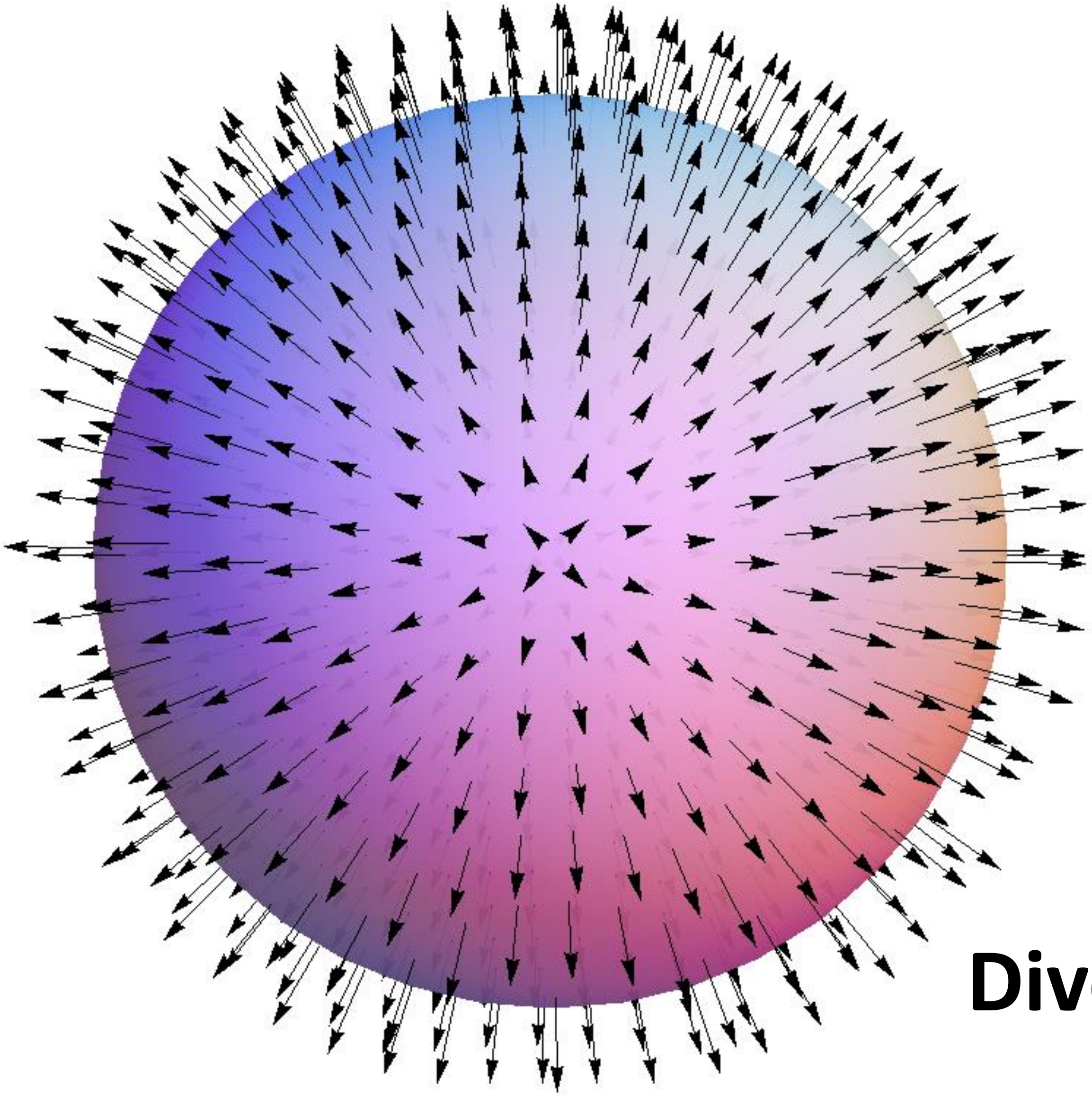
Given a vector $\vec{r} = r\hat{e_r}$, verify $\int_V \vec{\nabla} \cdot \vec{r} dV = \int_S \vec{r}.ds$, for a sphere of radius R. What do you physically infer from this?

In this problem we need to verify $\int_V \vec{\nabla} \cdot \vec{r} dV = \int_S \vec{r}.ds$, for a sphere of radius R, given $\vec{r} = r\hat{e_r}$.

$$\vec{\nabla} \bullet \vec{A} = \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(A_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \right)$$

$$\int_V \vec{\nabla} \cdot \vec{r} dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \times r) r^2 \sin\theta d\theta d\phi dr = 4\pi R^3$$

The surface integral of the same is given as
$\int_S \vec{r}.ds = \int_0^{2\pi} \int_0^{\pi} R\hat{e_r}.R^2 \sin\theta d\theta d\phi \hat{e_r} = 4\pi R^3$



The sum of diverging field
Lines coming from inside
Of the spherical volume

=

Sum of the field lines
In the surface of the sphere

Divergence Theorem

Curl in Spherical Polar coordinate system

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$\vec{\nabla} \times \vec{A} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right) \times \left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right)$$

After some steps and rearranging the terms

$$\vec{\nabla} \times \vec{A} = \frac{\hat{e}_r}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial \phi} \right) + \frac{\hat{e}_\theta}{r\sin\theta}\left(\frac{\partial A_r}{\partial \phi} - \sin\theta \frac{\partial\left(rA_\phi\right)}{\partial r} \right) + \frac{\hat{e}_\phi}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right)$$

Try by self

Laplacian – a scalar operator

$$\vec{\nabla}^2 f = \vec{\nabla} \bullet \left(\vec{\nabla} f \right)$$

Laplacian – In Cartesian coordinate system

$$\vec{\nabla}^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Laplacian – Spherical polar Coordinate system

$$\vec{\nabla}^2 f = \vec{\nabla}\cdot\left(\vec{\nabla} f\right)$$

$$\vec{\nabla}\cdot\left(\vec{\nabla} f\right) = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\cdot\left(\hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)$$

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}\right)$$

$$\vec{A} = \hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi$$

$$\vec{\nabla}^2 f = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}\right)$$

Divergence and curl in Cylindrical polar Coordinate system (Try yourself)

$$\vec{\nabla}\cdot\vec{A}=\left(\hat{e}_\rho\frac{\partial}{\partial\rho}+\hat{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi}+\hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{e}_\rho A_\rho+\hat{e}_\phi A_\phi+\hat{e}_z A_z\right)$$

$$\vec{\nabla}\bullet\vec{A}=\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\rho\right)+\frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}\right)$$

$$\vec{\nabla}\times\vec{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\hat{e}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\hat{e}_\phi+\frac{1}{\rho}\left(\frac{\partial\left(\rho A_\phi\right)_z}{\partial\rho}-\frac{\partial A_\rho}{\partial z}\right)\hat{e}_z$$

Easy way to remember

General Orthogonal Coordinates

Cartesian Coordinates

$$q_1 = x, \quad q_2 = y, \quad q_3 = z; \quad h_1 = h_2 = h_3 = 1,$$

Cylindrical Coordinates

$$q_1 = \rho, \quad q_2 = \varphi, \quad q_3 = z; \quad h_1 = h_\rho = 1, \quad h_2 = h_\varphi = \rho, \quad h_3 = h_z = 1,$$

Spherical Polar Coordinates

$$q_1 = r, \quad q_2 = \theta, \quad q_3 = \varphi; \quad h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\varphi = r\sin\theta,$$

$$\nabla V = \sum_i \frac{1}{h_i}\frac{\partial V}{\partial q_i}\hat{\mathbf{q}}_i$$

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(F_1 h_2 h_3) + \frac{\partial}{\partial q_2}(F_2 h_3 h_1) + \frac{\partial}{\partial q_3}(F_3 h_1 h_2)\right]$$

$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial q_3}\right)\right]$$

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{q}}_1 & h_2\hat{\mathbf{q}}_2 & h_3\hat{\mathbf{q}}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

Examples

$$\vec{\nabla}\times\vec{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix} = \frac{\widehat{e}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\phi\sin\theta\right) - \frac{\partial F_\theta}{\partial\phi}\right) +$$

$$\frac{\widehat{e}_\theta}{r\sin\theta}\left(\frac{\partial F_r}{\partial\phi} - \sin\theta\frac{\partial\left(rF_\phi\right)}{\partial r}\right) +$$

$$\frac{\widehat{e}_\phi}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)$$

$$\vec{\nabla} \times \vec{F} = \frac{\hat{e}_r}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial\phi}\right) + \frac{\hat{e}_\theta}{r\sin\theta}\left(\frac{\partial F_r}{\partial\phi} - \sin\theta\frac{\partial\left(rF_\phi\right)}{\partial r}\right) + \frac{\hat{e}_\phi}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right) = 0$$