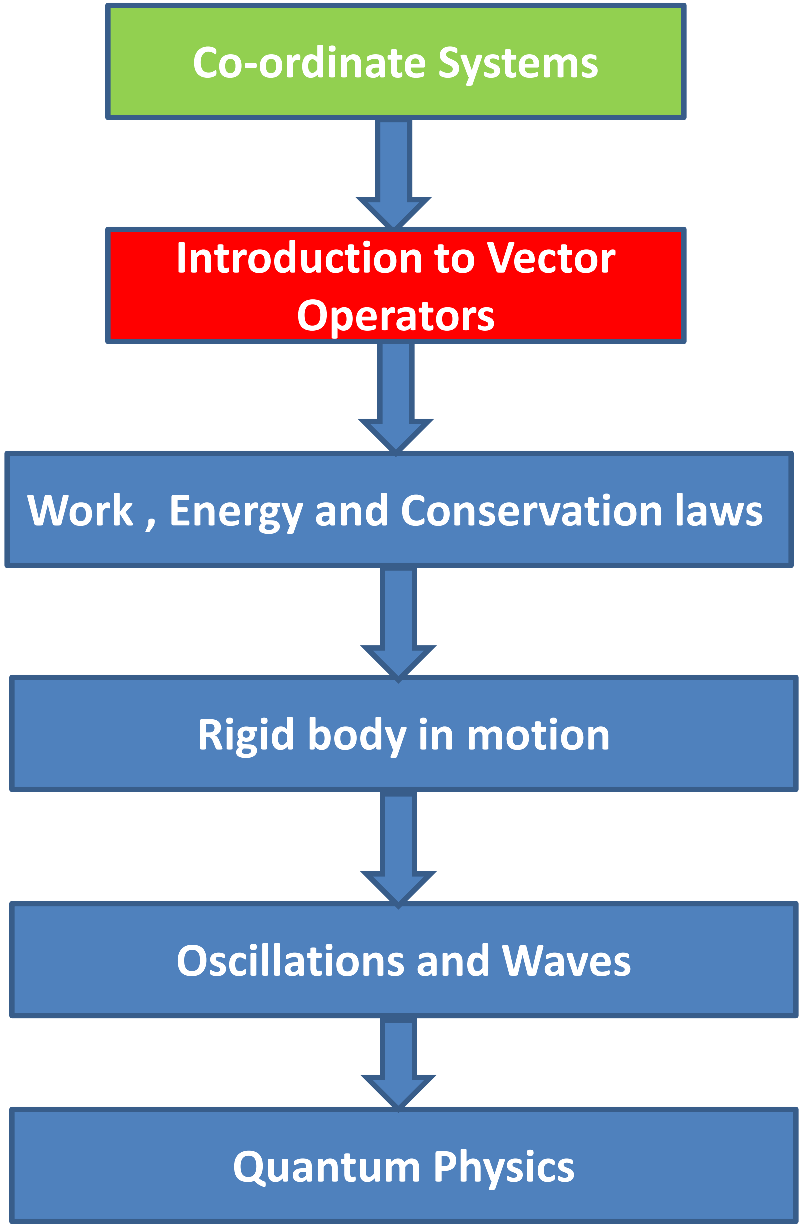
Highlights of the course



Chapter-2: Introduction to Vector Operators

Gradient, Divergence and Curl

$\vec{\nabla} \bullet \vec{V}$ (The divergence)

$\vec{\nabla}.\vec{V}$ (The divergence)

$$\vec{\nabla}.\vec{V} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right) \bullet \left(\hat{e}_x V_x + \hat{e}_y V_y + \hat{e}_z V_z \right)$$

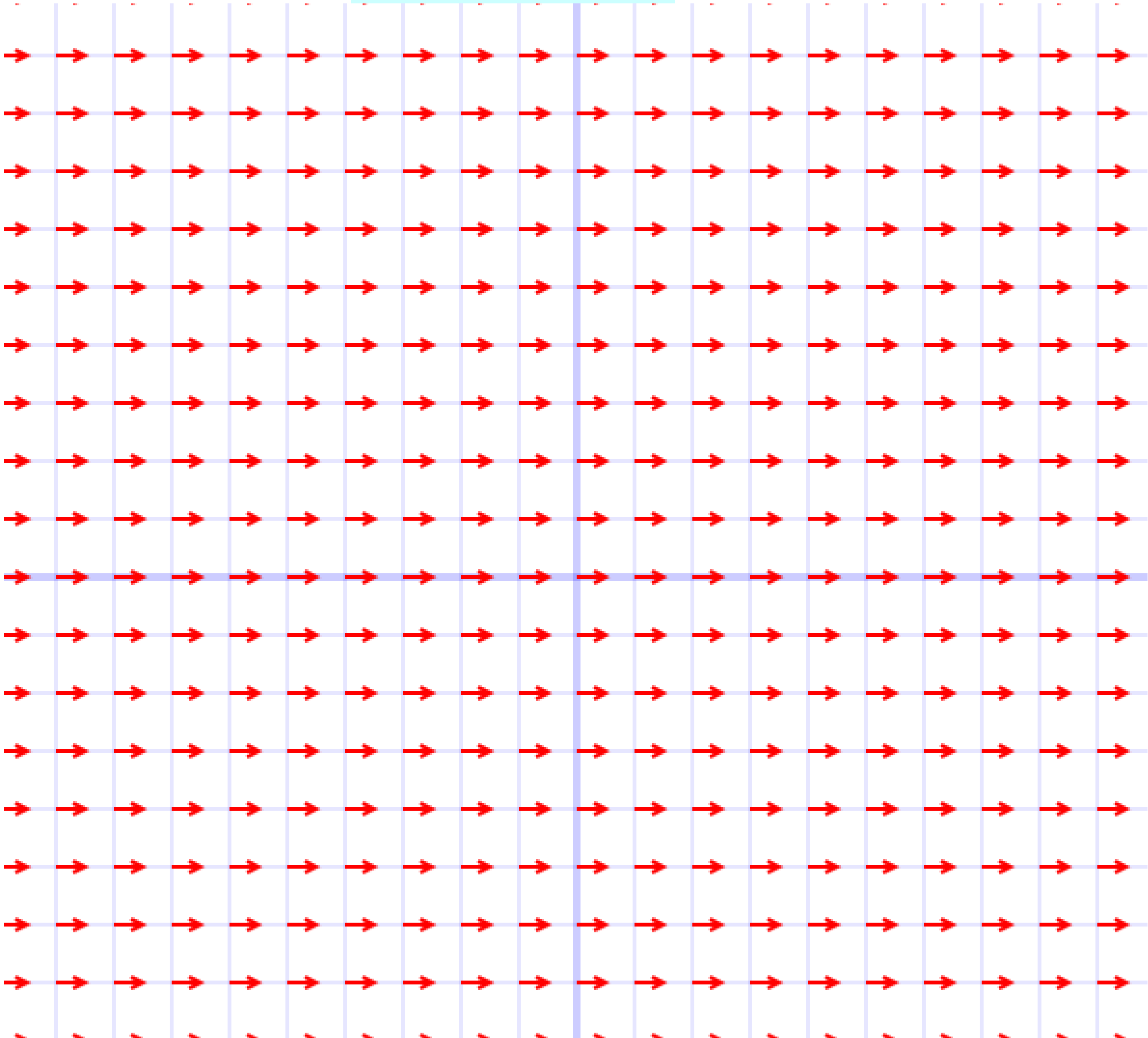
$$\vec{\nabla}.\vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right)$$

The divergence: Geometrical interpretation

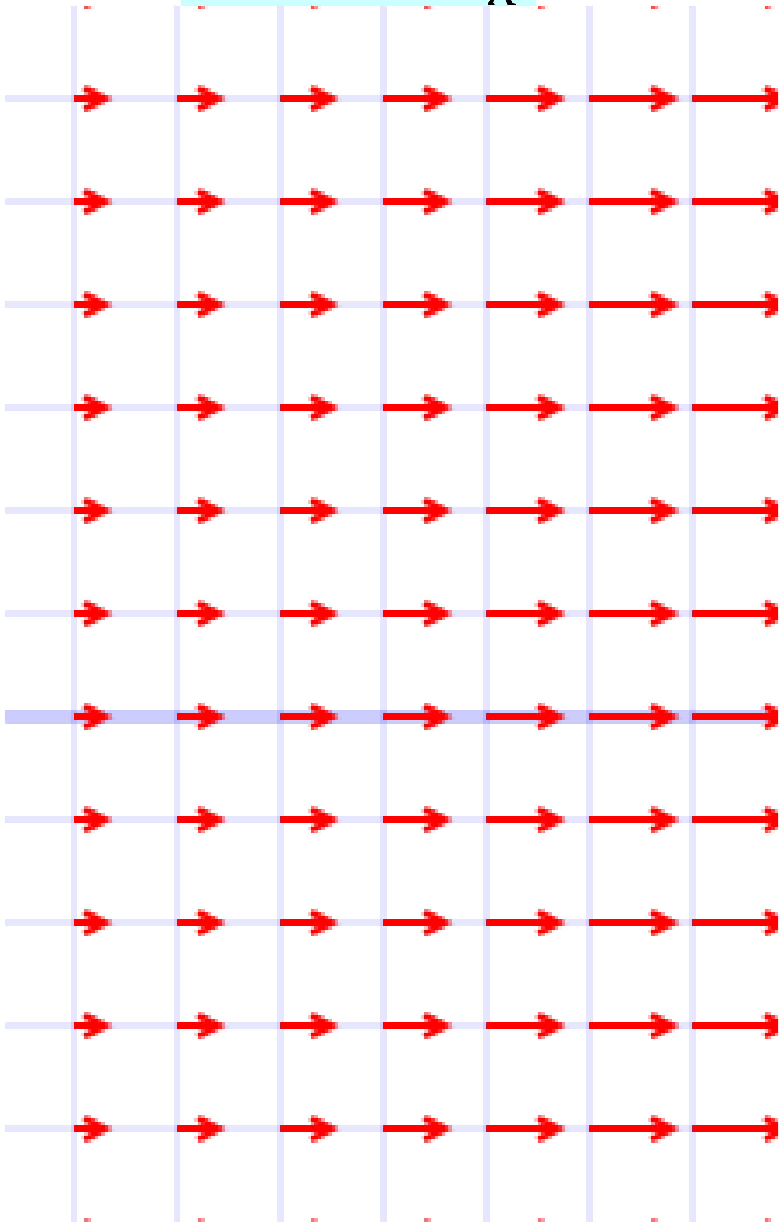
Net outward/inward flow

Will try to understand it through some examples

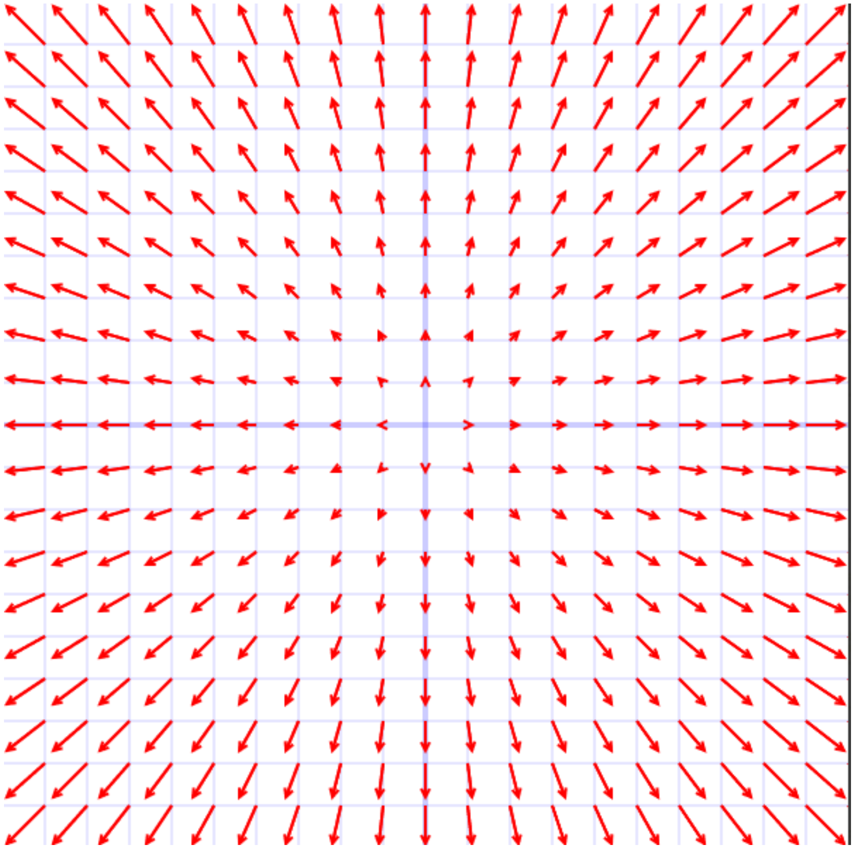
$$\vec{A} = 1\hat{e}_x$$



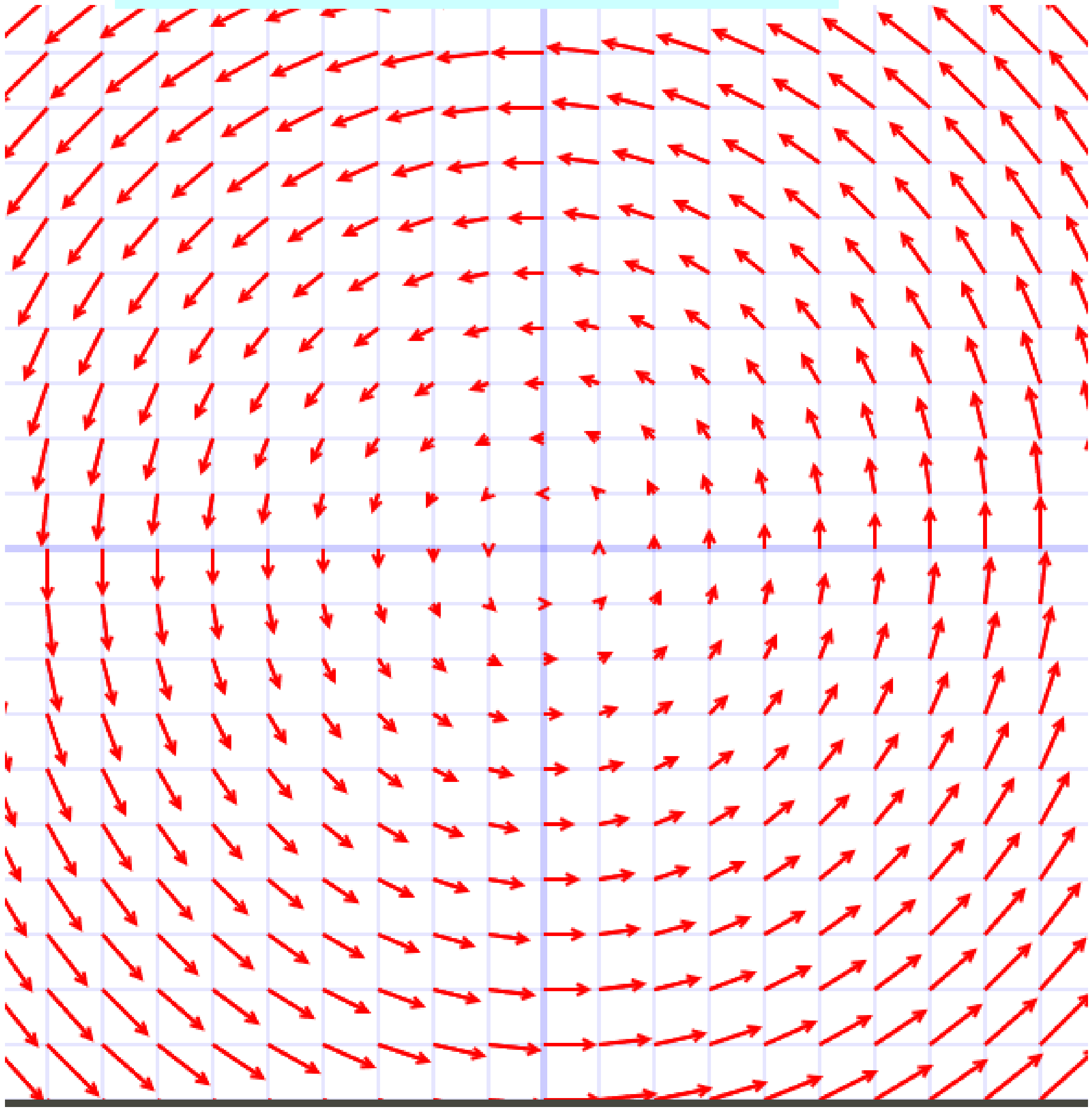
$$\vec{A} = x\hat{e}_x$$



$$\overrightarrow{A} = x\hat{e}_x + y\hat{e}_y$$



$$\overrightarrow{A} = \hat{e}_x + \hat{e}_y$$



$\vec{\nabla}.\vec{V}$ (The divergence)

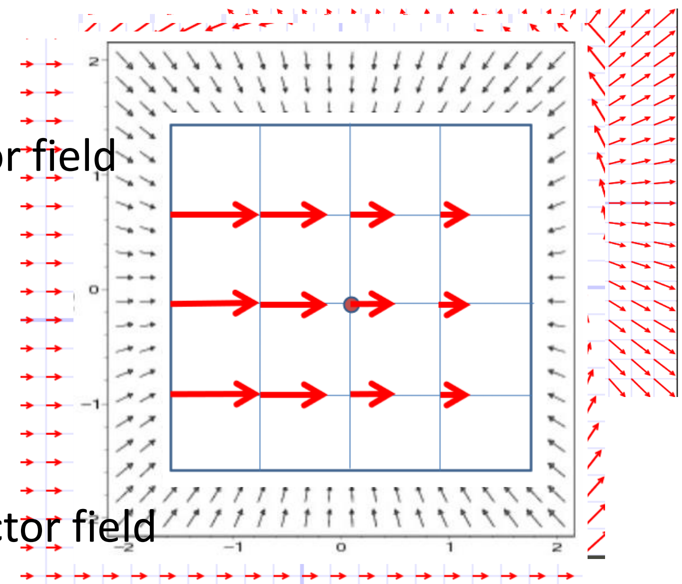
$$\vec{\nabla}.\vec{V} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right) \bullet \left(\hat{e}_x V_x + \hat{e}_y V_y + \hat{e}_z V_z \right)$$

$$\vec{\nabla}.\vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right)$$

Positive Divergence indicates net increase in the flux of Vector field

0 Divergence indicates constant flow

Negative Divergence indicates net decrease in the flux of Vector field



$$\vec{\nabla} \times \vec{V} \text{ (The curl)}$$

Measure of rotation

$\vec{\nabla} \times \vec{V}$ (The curl)

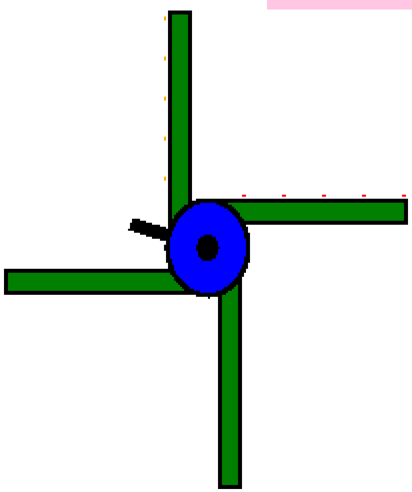
$$\vec{\nabla} \times \vec{V} = \hat{e}_x \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{e}_y \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{e}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

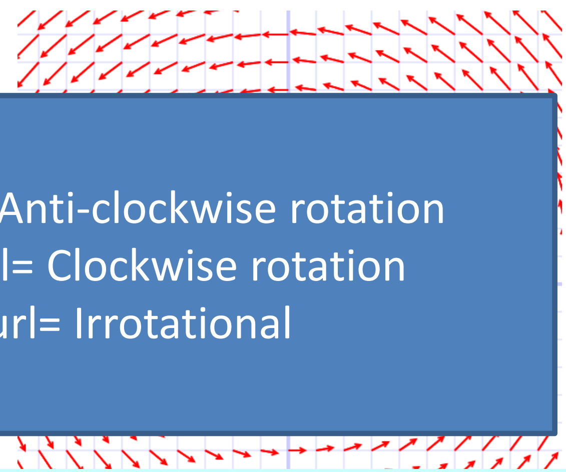
The curl: Geometrical interpretation

Measure of rotation

$$\vec{V} = V_x(x,y)\hat{e}_x + V_y(x,y)\hat{e}_y :$$

$$\vec{\nabla} \times \vec{V} = \hat{e}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$





Positive Curl = Anti-clockwise rotation
Negative Curl= Clockwise rotation
Zero Curl= Irrotational

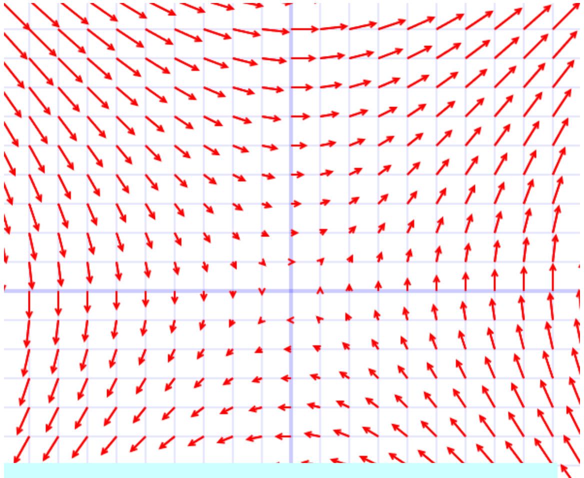
$$\vec{V} = -y\hat{e}_x + x\hat{e}_y :$$

$\vec{\nabla} \times \vec{V}$ (The curl)

$$\vec{\nabla} \times \vec{V} = \hat{e}_x \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{e}_y \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{e}_z \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

The curl: Geometrical interpretation

Measure of rotation



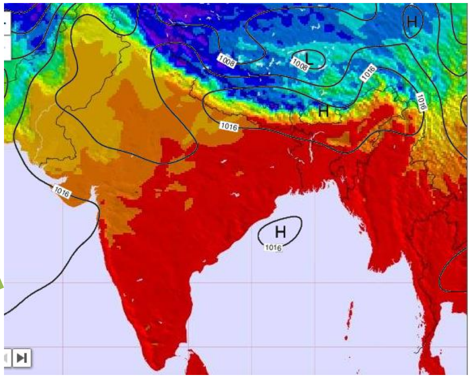
$$\vec{V} = y\hat{e}_x + x\hat{e}_y :$$

Gradient of a scalar function in different coordinate system

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$dT = \vec{\nabla}T \bullet d\vec{l}$$

This definition is independent of the coordinate system used



$$T(x, y, z)$$

$$T(r, \theta, \phi)$$

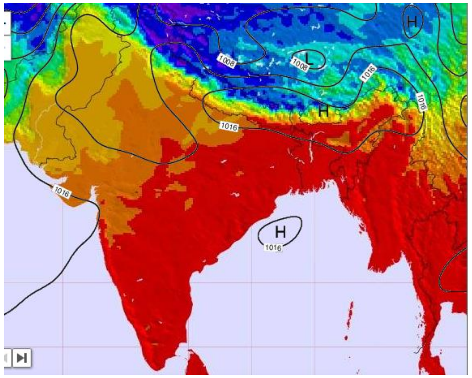
$$T(\rho, \phi, z)$$

$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \phi}d\phi$$

$$dT = \frac{\partial T}{\partial \rho}d\rho + \frac{\partial T}{\partial \phi}d\phi + \frac{\partial T}{\partial z}dz$$

Gradient in cylindrical polar coordinate system

$$dT = \frac{\partial T}{\partial \rho} d\rho + \frac{\partial T}{\partial \phi} d\phi + \frac{\partial T}{\partial z} dz$$



$$T(\rho, \phi, z)$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$

$$d\vec{l} = d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z$$

$$dT = \left(\vec{\nabla} T\right) . \left(d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z\right)$$

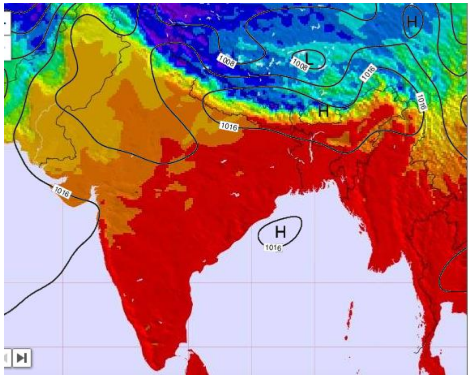
$$dT = \frac{\partial T}{\partial \rho} d\rho + \frac{\partial T}{\partial \phi} d\phi + \frac{\partial T}{\partial z} dz \overset{?}{=} \left(\vec{\nabla} T\right) . \left(d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z\right)$$

$$\vec{\nabla} T = \left(\frac{\partial T}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \hat{e}_\phi + \frac{\partial T}{\partial z} \hat{e}_z \right)$$

Gradient

Gradient in cylindrical polar coordinate system

$$dT = \frac{\partial T}{\partial \rho} d\rho + \frac{\partial T}{\partial \phi} d\phi + \frac{\partial T}{\partial z} dz$$



$$T(\rho, \phi, z)$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$

$$\vec{\nabla} T = \left(\frac{\partial T}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \hat{e}_\phi + \frac{\partial T}{\partial z} \hat{e}_z \right)$$

Gradient

$$dT = \left(\frac{\partial T}{\partial \rho} \hat{e}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \hat{e}_\phi + \frac{\partial T}{\partial z} \hat{e}_z \right) . \left(d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z \right)$$

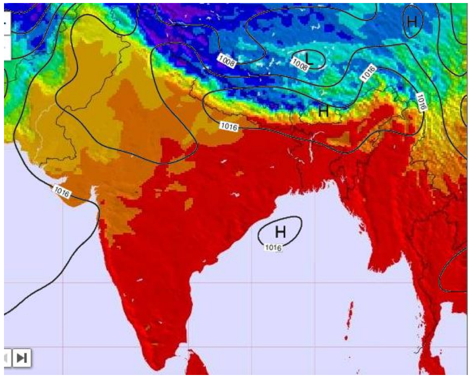
$$\vec{\nabla} = \left(\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Gradient in spherical polar coordinate system

$$dT = \frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial \theta} d\theta + \frac{\partial T}{\partial \phi} d\phi$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$

$$d\vec{l} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$$



$T(r,\theta,\phi)$

$$dT = \left(\vec{\nabla} T\right).\left(dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi\right)$$

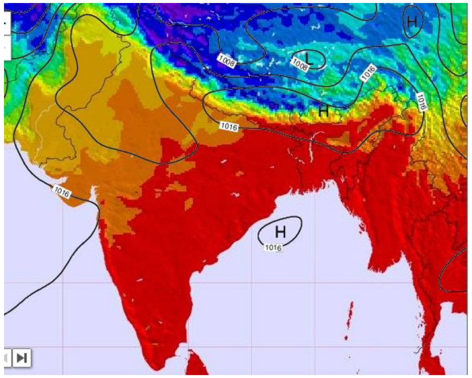
$$\vec{\nabla} T = \left(\frac{\partial T}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{e}_\phi\right)$$

Gradient

Gradient in spherical polar coordinate system

$$dT = \frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial \theta} d\theta + \frac{\partial T}{\partial \phi} d\phi$$

$$dT = \vec{\nabla} T \bullet d\vec{l}$$



$$T(r, \theta, \phi)$$

$$dT = \left(\frac{\partial T}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi} \hat{e}_\phi \right) . \left(dr \hat{e}_r + r d\theta \hat{e}_\theta + r \sin\theta d\phi \hat{e}_\phi \right)$$

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

Del operator in different coordinate system

Cartesian Coordinate system

$$\vec{\nabla} = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Cylindrical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z} \right)$$

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right)$$

Divergence in Spherical Polar coordinate system

Spherical polar Coordinate system

$$\vec{\nabla} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \right)$$

$$\vec{A} = \hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi$$

$$\vec{\nabla}.\vec{A} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi} \right).\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right)$$

$$\hat{e}_r . \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi \right) +$$

Divergence in Spherical polar Coordinate system

$$\hat{e}_r \cdot \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\phi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right)$$

$$\hat{e}_r \cdot \left(\hat{e}_r \frac{\partial A_r}{\partial r} + A_r \frac{\partial \hat{e}_r}{\partial r}\right) +$$

Derivatives of Unit Vectors

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\frac{d\hat{e}_r}{dr} = 0 \qquad \frac{d\hat{e}_\theta}{dr} = 0 \qquad \frac{d\hat{e}_\phi}{dr} = 0$$

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta \qquad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r \qquad \frac{d\hat{e}_\phi}{d\theta} = 0$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi \qquad \frac{d\hat{e}_\theta}{d\phi} = \cos\theta\hat{e}_\phi \qquad \frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

Divergence in Spherical polar Coordinate system

$$\hat{e}_r \cdot \frac{\partial}{\partial r}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right) +$$

$$\hat{e}_\phi \cdot \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi\right)$$

$$\hat{e}_r \cdot \left(\hat{e}_r \frac{\partial A_r}{\partial r} + A_r \frac{\partial \hat{e}_r}{\partial r}\right) +$$

$$\hat{e}_r \cdot \left(\hat{e}_\theta \frac{\partial A_\theta}{\partial r} + A_\theta \frac{\partial \hat{e}_\theta}{\partial r}\right) +$$

$$\hat{e}_r \cdot \left(\hat{e}_\phi \frac{\partial A_\phi}{\partial r} + A_\phi \frac{\partial \hat{e}_\phi}{\partial r}\right)$$

and proceeding further.....

Divergence in Spherical polar Coordinate system

$$\vec{\nabla}.\vec{A} = \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \right)$$