1. Co-ordinate Systems Continued… (Lecture 4)

Choice of Co-ordinate systems

Cartesian Co-ordinates: Line, area and volume element

Plane-Polar Co-ordinates: Unit vectors, transformations, Rate of change, velocity and acceleration, Line element, area element

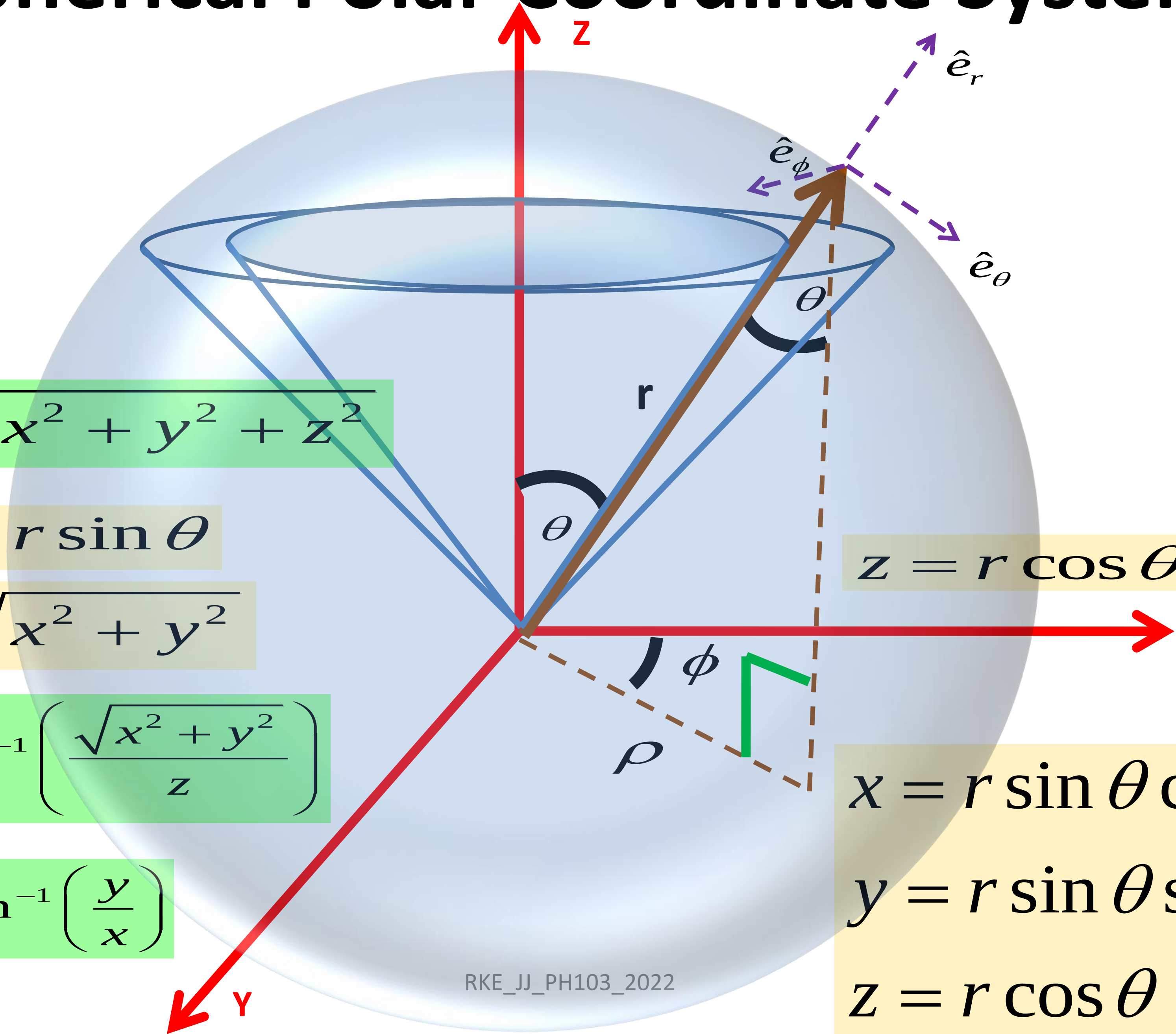
Cylindrical Co-ordinates: Unit vectors and its transformations, Rate of change, velocity and acceleration, line, area and volume element

Lec 1-3 & Tut1

Spherical Polar Co-ordinates: Unit vectors and its transformations, Rate of change, velocity and acceleration, line, area and volume element

Plane polar	Cylindrical polar
(r,θ)	(ρ,ϕ,z)
$\vec{r}=r\hat{e}_r$	$\vec{r}=\rho\hat{e}_\rho+z\hat{e}_z$
$d\vec{r}=dr\hat{e}_r+rd\theta\hat{e}_\theta$	$d\vec{r}=d\rho\hat{e}_\rho+\rho d\phi\hat{e}_\phi+dz\hat{e}_z$
$\hat{e}_r=\hat{e}_x\cos(\theta)+\hat{e}_y\sin(\theta)$ $\hat{e}_\theta=-\hat{e}_x\sin(\theta)+\hat{e}_y\cos(\theta)$	$\hat{e}_\rho=\hat{e}_x\cos(\phi)+\hat{e}_y\sin(\phi)+0\hat{e}_z$ $\hat{e}_\phi=-\hat{e}_x\sin(\phi)+\hat{e}_y\cos(\phi)+0\hat{e}_z$ $\hat{e}_z=\hat{e}_z$
$\vec{V}=\dot{r}\,\hat{e}_r+r\dot{\theta}\,\hat{e}_\theta$	$\vec{V}=\dot{\rho}\,\hat{e}_\rho+\rho\dot{\phi}\,\hat{e}_\phi+\dot{z}\,\hat{e}_z$
$\vec{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{e}_\theta$	$\vec{a}=\left(\ddot{\rho}-\rho\dot{\phi}^2\right)\hat{e}_\rho+\left(\rho\ddot{\phi}+2\dot{\rho}\dot{\phi}\right)\hat{e}_\phi+\ddot{z}\,\hat{e}_z$

Spherical Polar Coordinate System



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\rho = r\sin\theta$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = r\cos\theta$$

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

Transformation of Coordinates

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$$

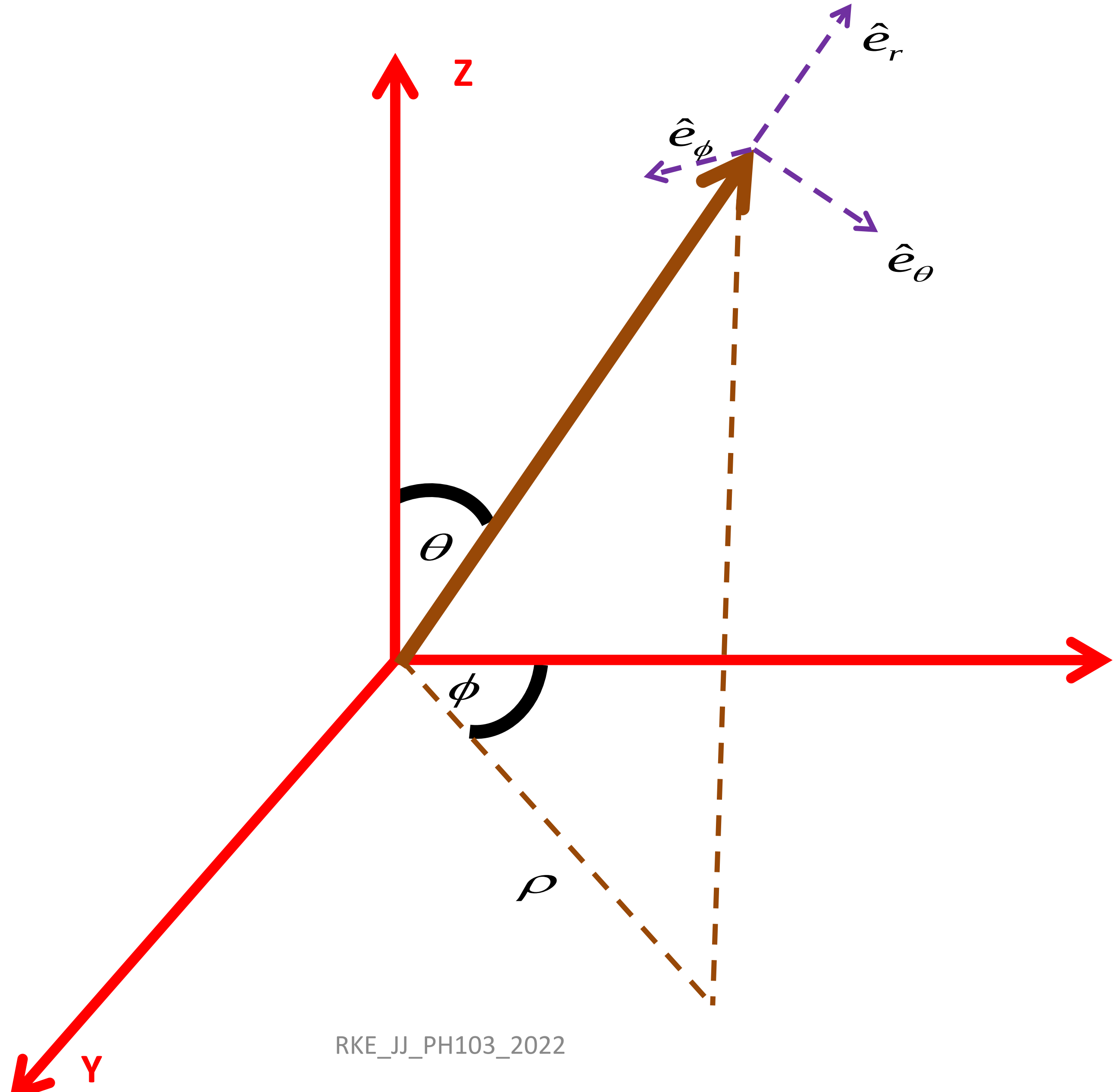
$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$x = r\sin\theta\cos\phi$$

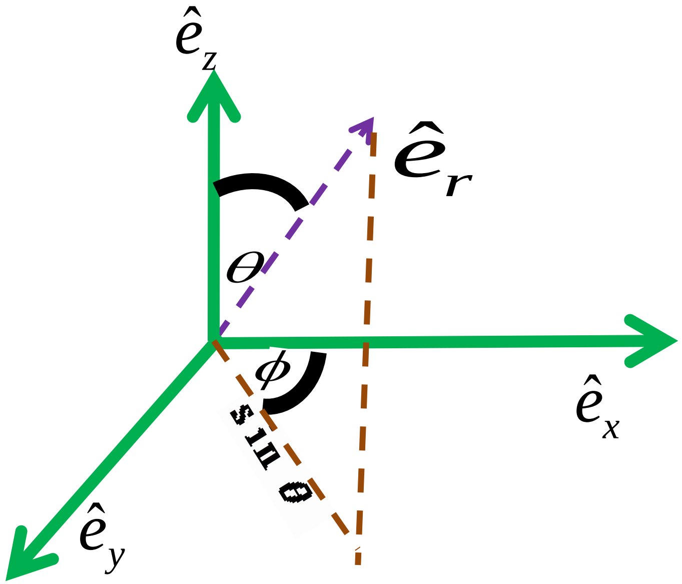
$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

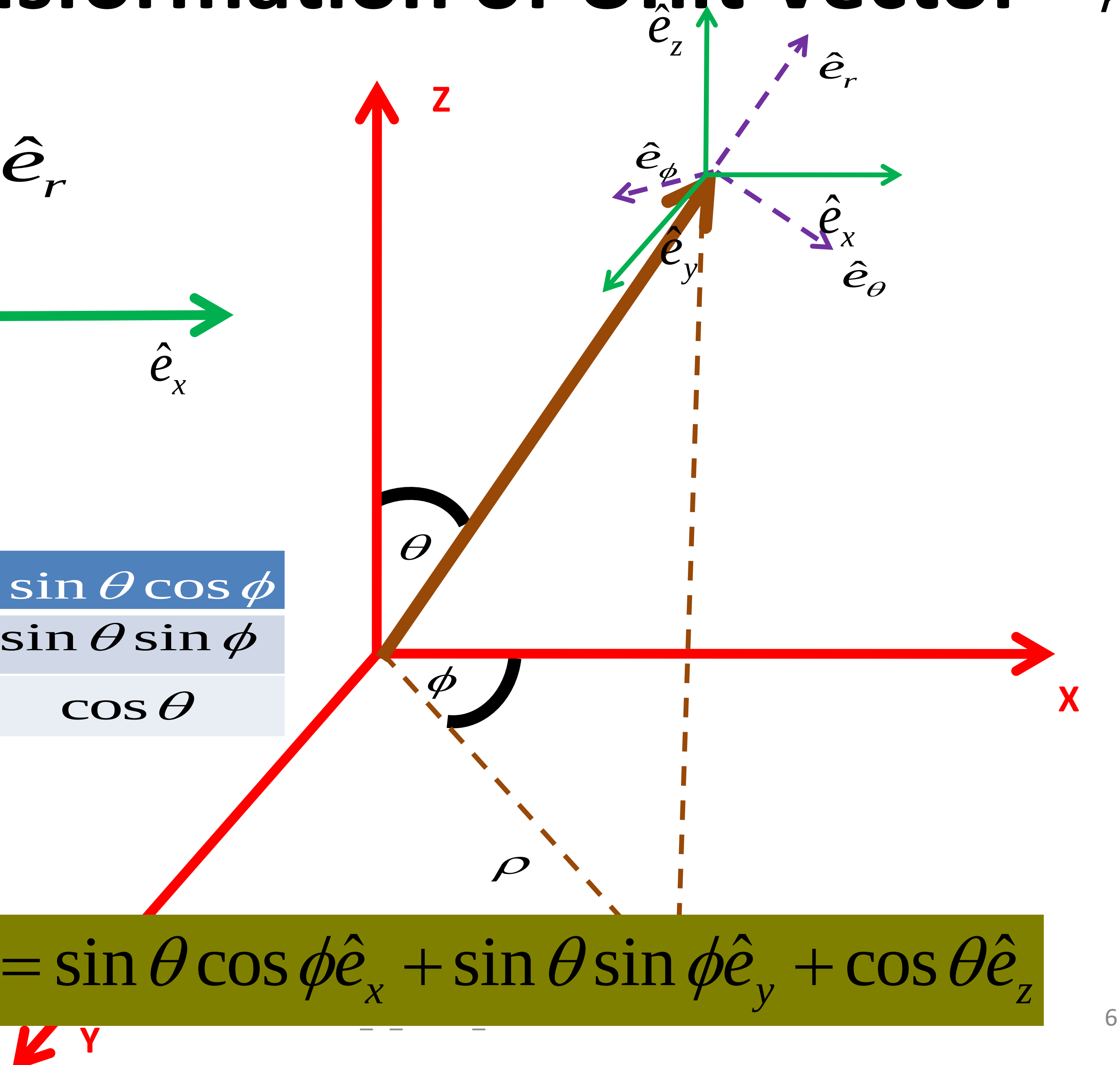
Transformation of Unit Vectors



Transformation of Unit Vector $\hat{e}_r$

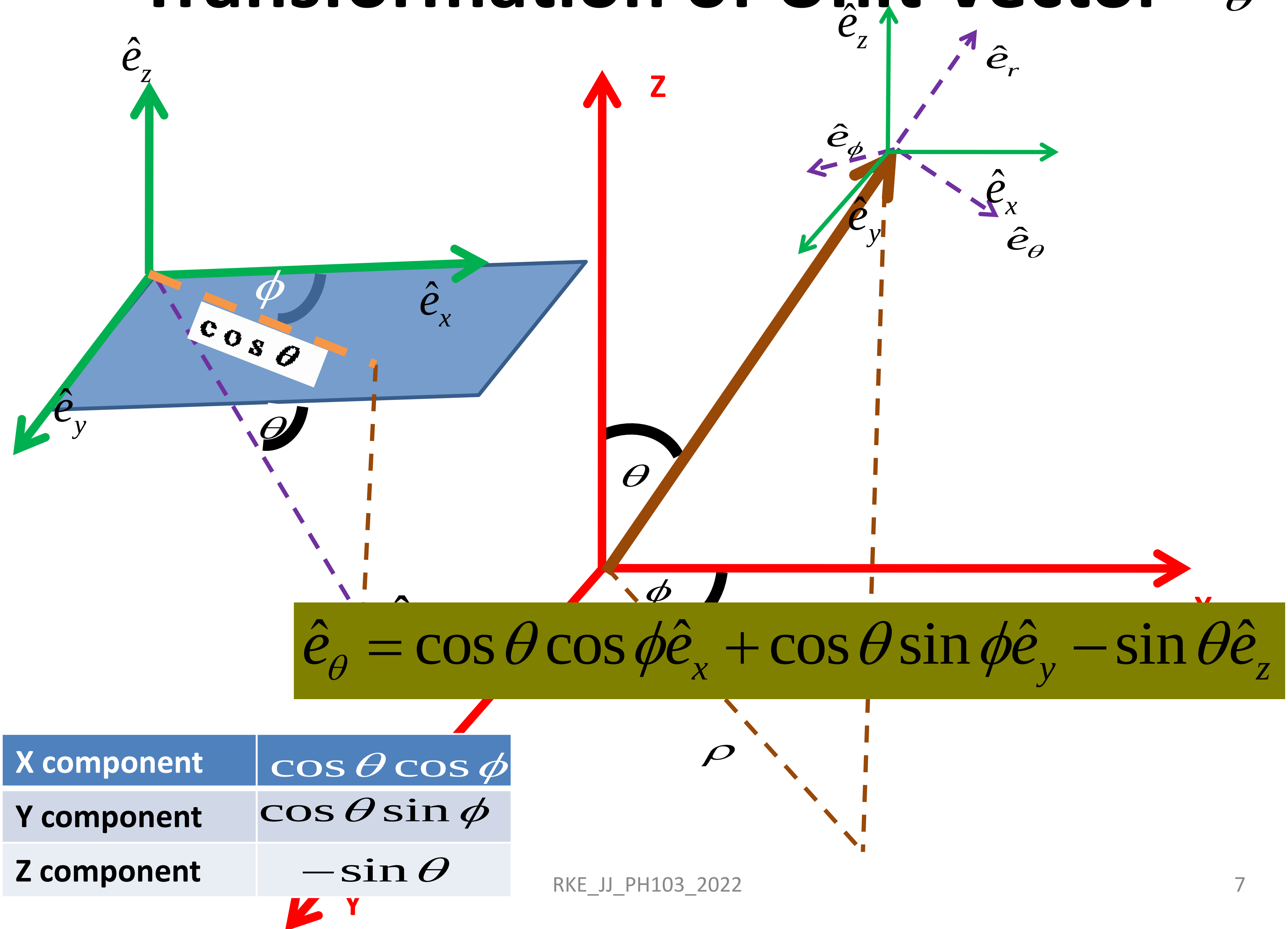


X component	$\sin\theta\cos\phi$
Y component	$\sin\theta\sin\phi$
Z component	$\cos\theta$



$$\hat{e}_r = \sin\theta\cos\phi\,\hat{e}_x + \sin\theta\sin\phi\,\hat{e}_y + \cos\theta\,\hat{e}_z$$

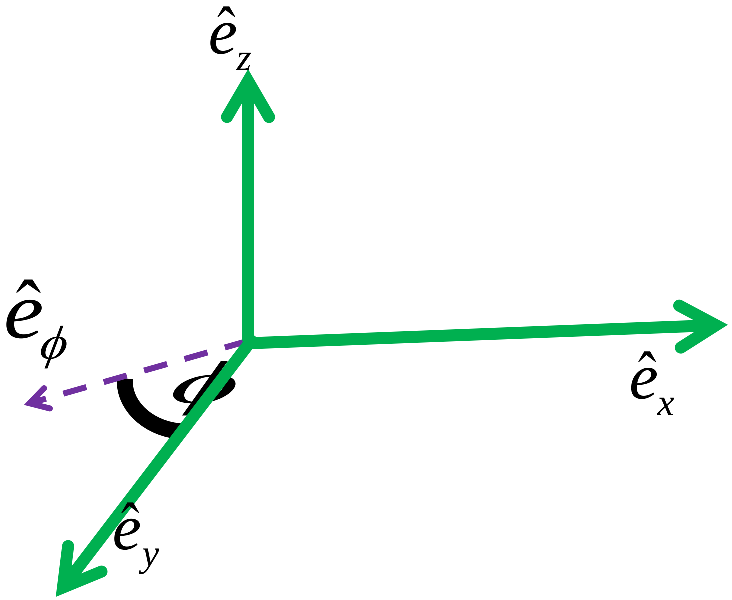
Transformation of Unit Vector $\hat{e}_\theta$

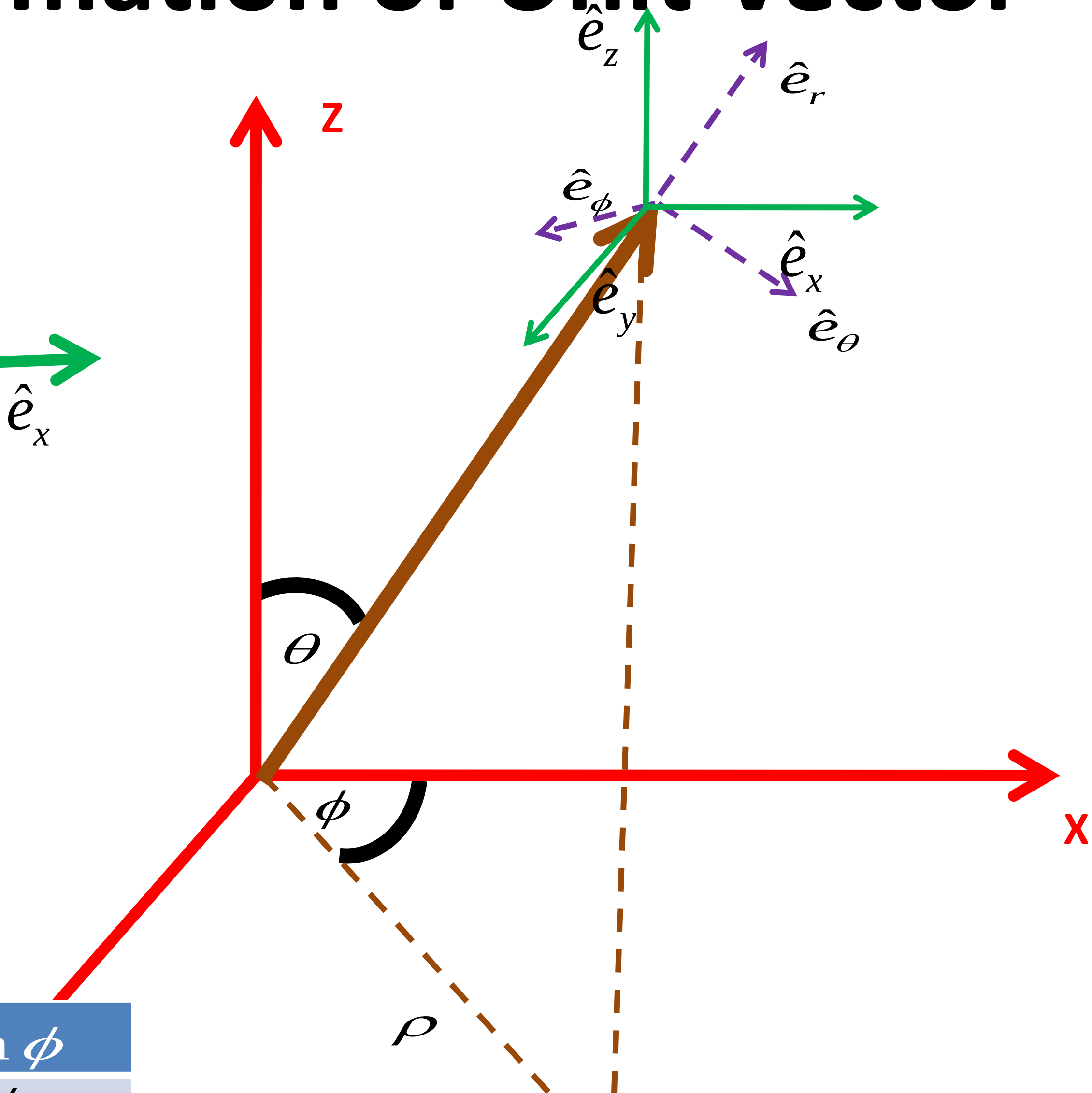


$$\hat{e}_\theta = \cos\theta\cos\phi\hat{e}_x + \cos\theta\sin\phi\hat{e}_y - \sin\theta\hat{e}_z$$

X component	$\cos\theta\cos\phi$
Y component	$\cos\theta\sin\phi$
Z component	$-\sin\theta$

Transformation of Unit Vector $\hat{e}_\phi$





X component	$-\sin\phi$
Y component	$\cos\phi$
Z component	0

$$\hat{e}_\phi = -\sin\phi\,\hat{e}_x + \cos\phi\,\hat{e}_y + 0\hat{e}_z$$

Transformation of Unit Vectors

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix}$$

Derivatives of Unit Vectors

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\frac{d\hat{e}_r}{dr} = 0$$

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi$$

$$\frac{d\hat{e}_\theta}{dr} = 0$$

$$\frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$

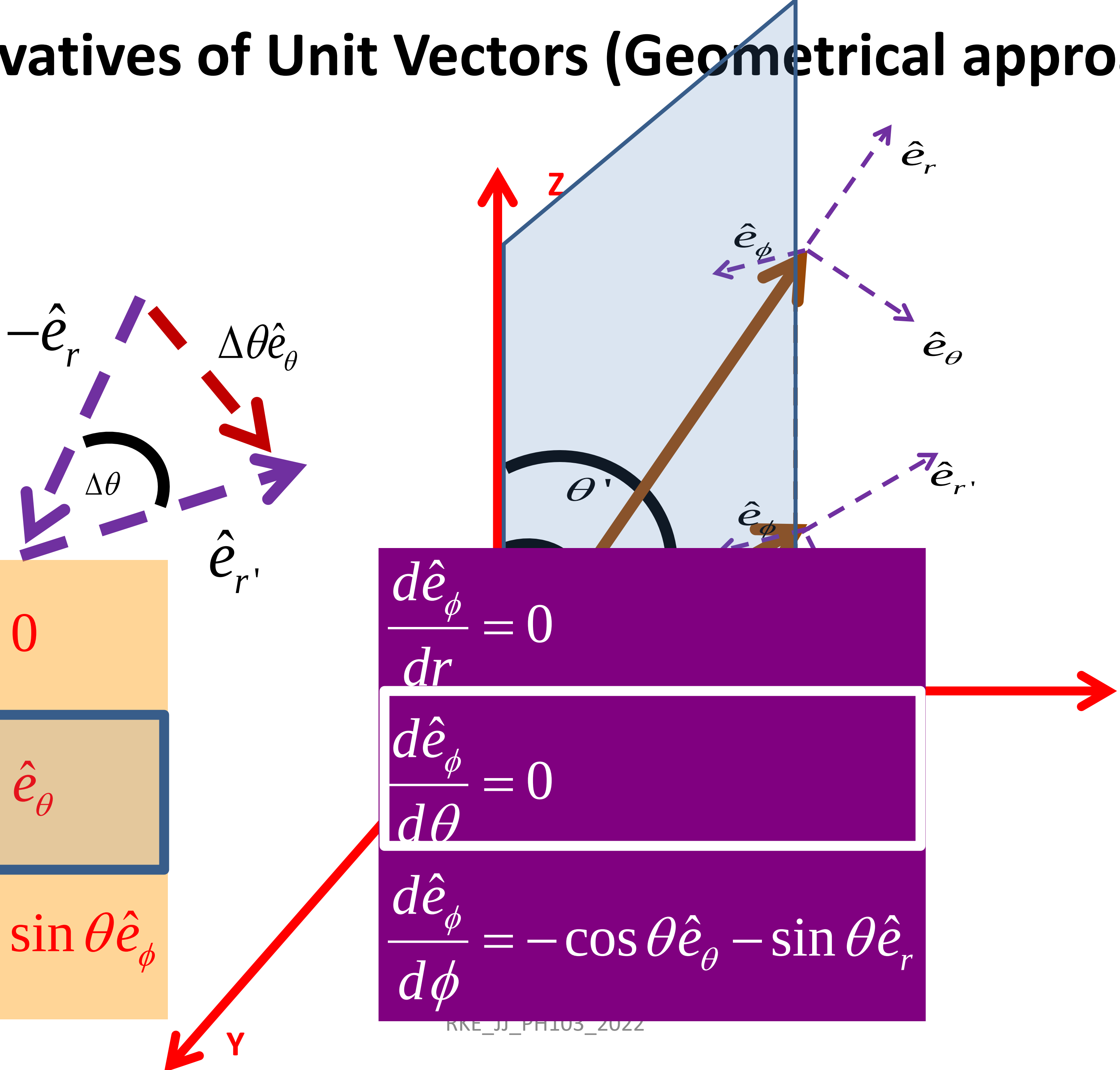
$$\frac{d\hat{e}_\theta}{d\phi} = \cos\theta\hat{e}_\phi$$

$$\frac{d\hat{e}_\phi}{dr} = 0$$

$$\frac{d\hat{e}_\phi}{d\theta} = 0$$

$$\frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

Derivatives of Unit Vectors (Geometrical approach)



$$\frac{d\hat{e}_r}{dr} = 0$$

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$$

$$\frac{d\hat{e}_r}{d\phi} = \sin\theta\hat{e}_\phi$$

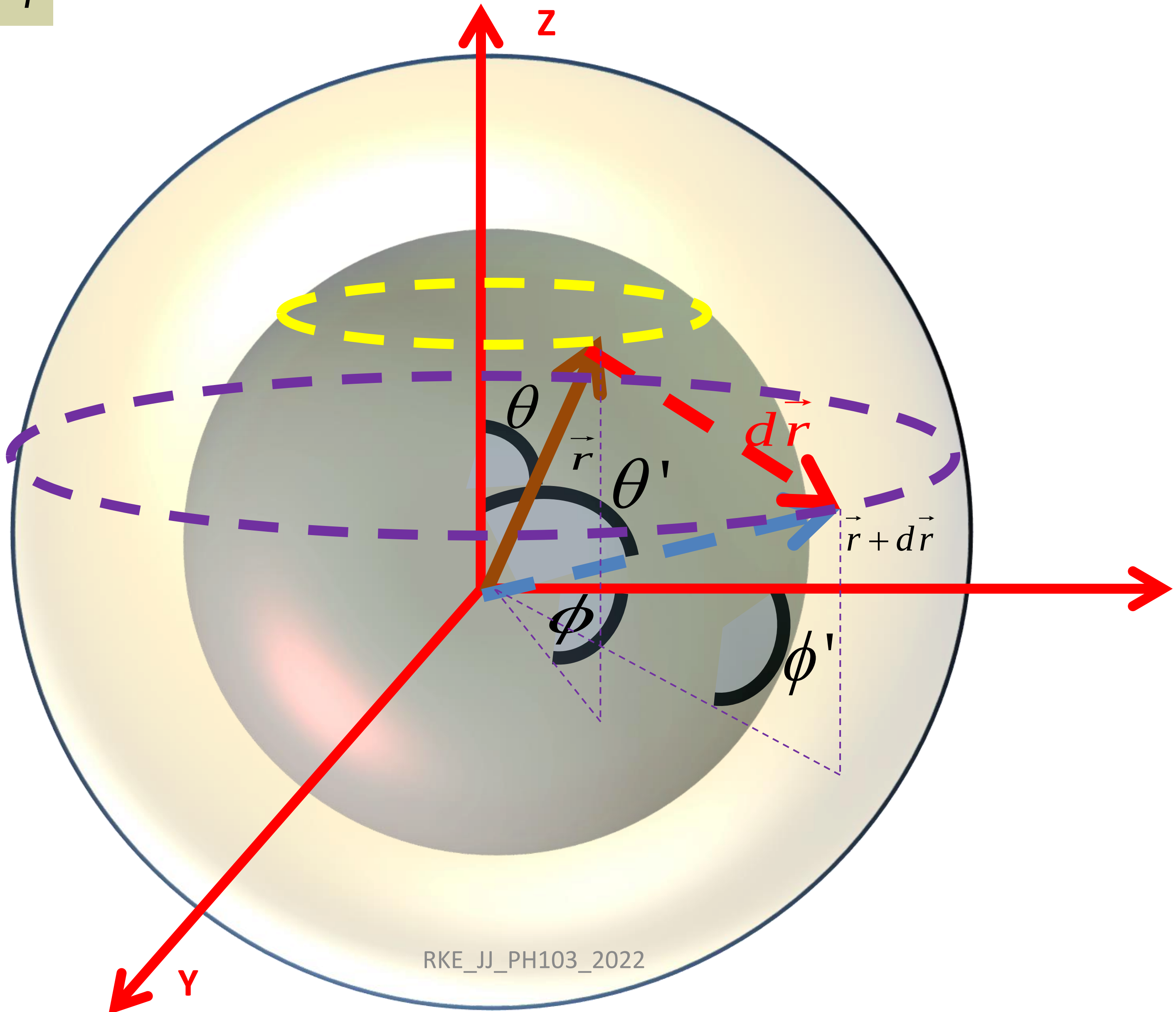
$$\frac{d\hat{e}_\phi}{dr} = 0$$

$$\frac{d\hat{e}_\phi}{d\theta} = 0$$

$$\frac{d\hat{e}_\phi}{d\phi} = -\cos\theta\hat{e}_\theta - \sin\theta\hat{e}_r$$

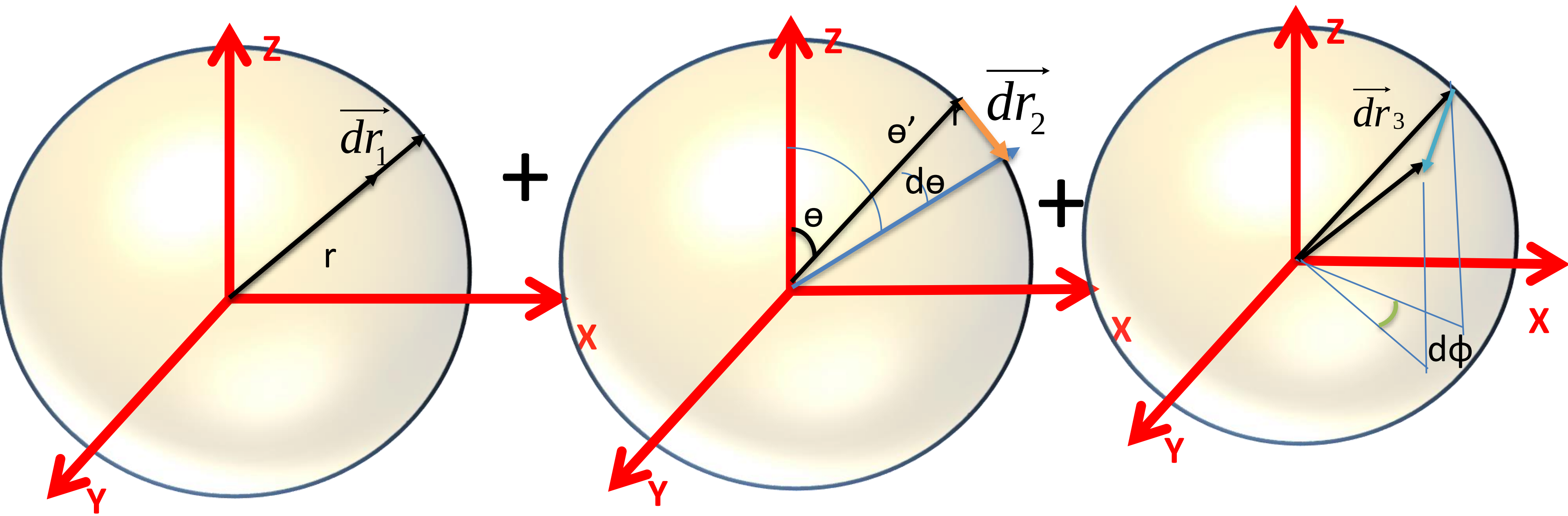
Infinitesimal line element

$$\vec{r} = r\hat{e}_r$$



How to find the line element?

Independent variation's of dr in r, ϴ and ϕ direction



Variation dr_1 in $\hat{e}_r$ direction
Keeping ϴ and ϕ constant

Variation dr_2 in $\hat{e}_\theta$ direction
Keeping r and ϕ constant

Variation dr_3 in $\hat{e}_\phi$ direction
Keeping r and ϴ constant

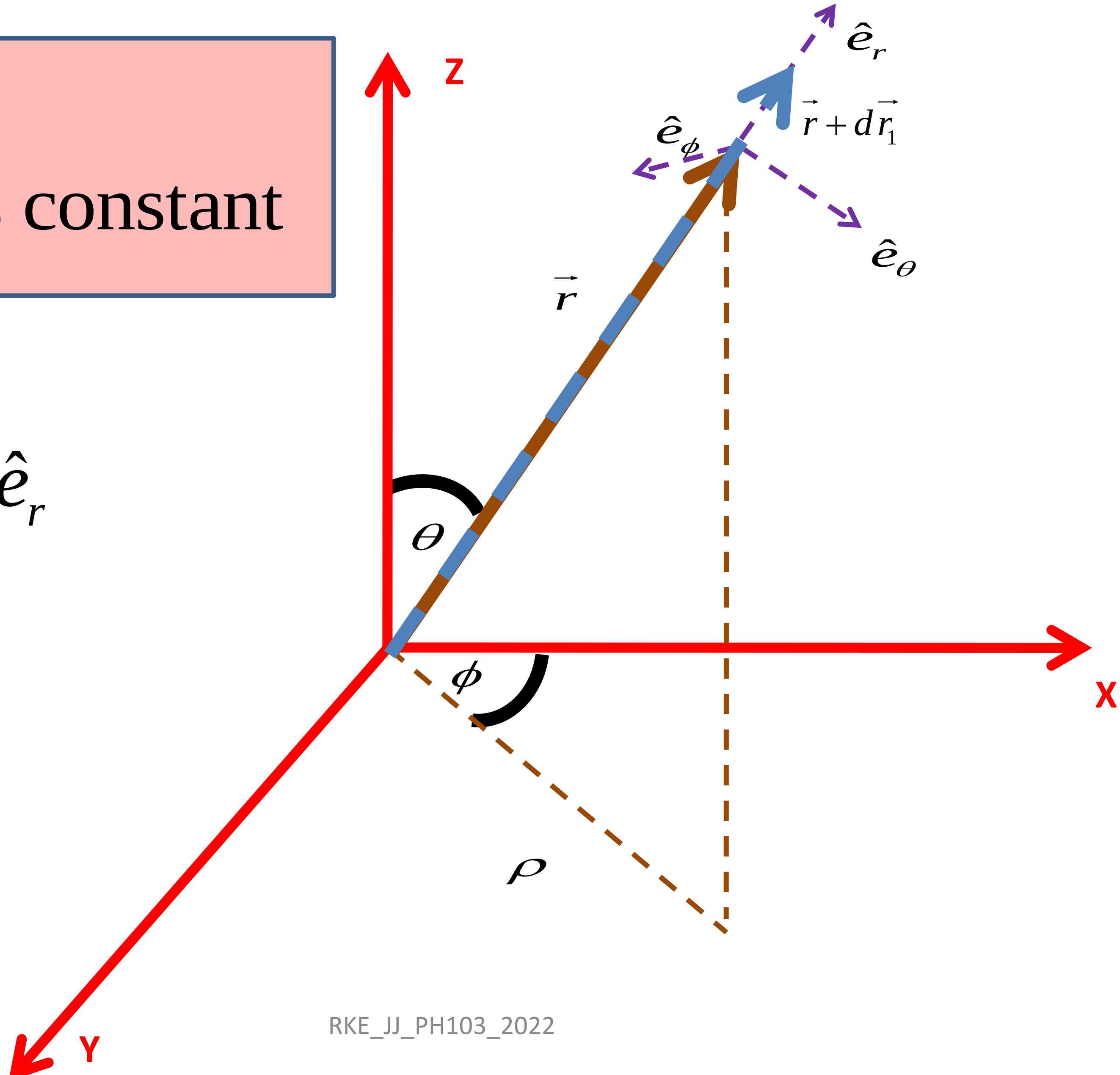
$$\overrightarrow{dr} = \overrightarrow{dr_1} + \overrightarrow{dr_2} + \overrightarrow{dr_3}$$

Infinitesimal line element

Case 1:

θ and ϕ is constant

$$d\vec{r_1} = dr\hat{e}_r$$

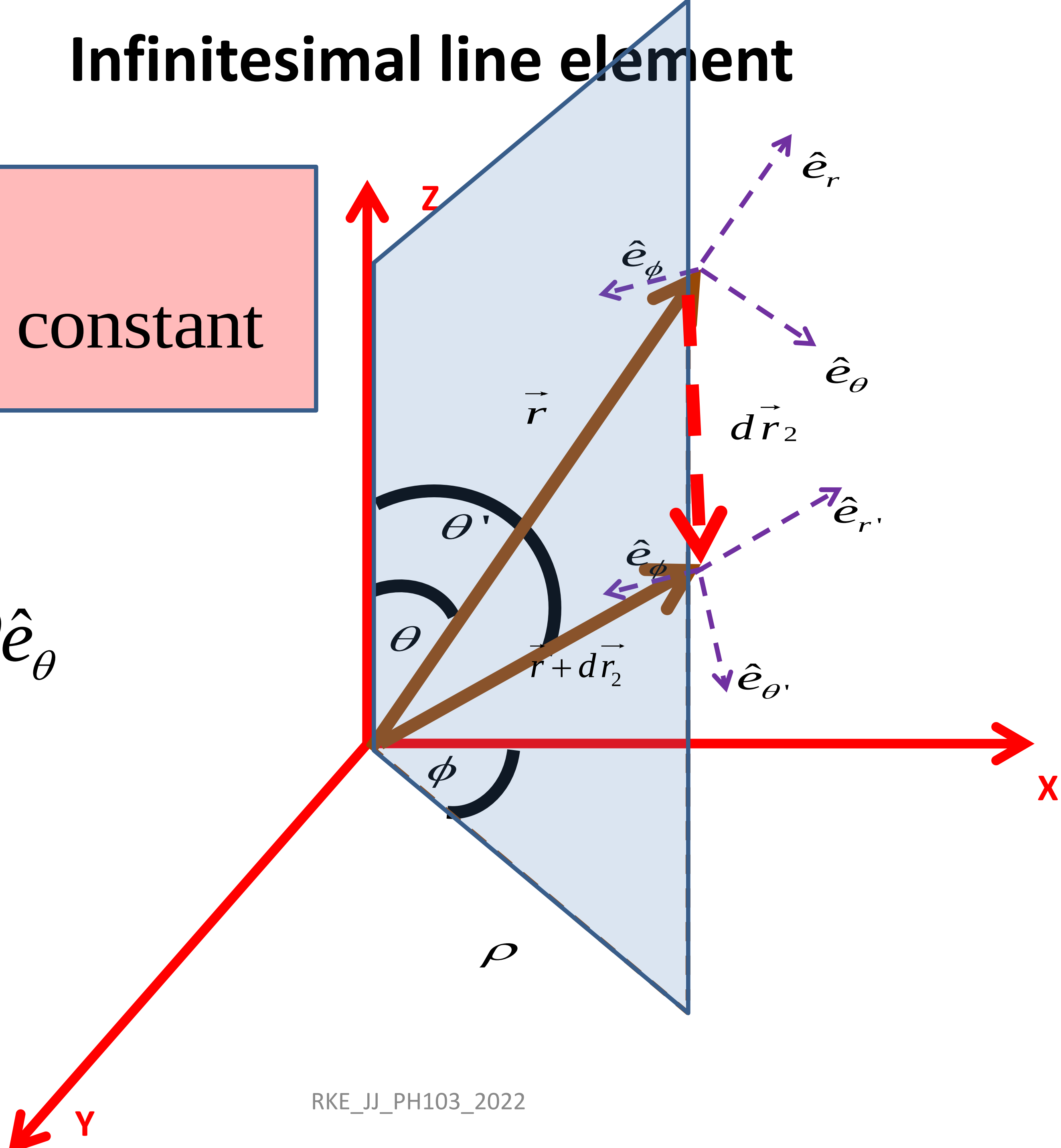


Infinitesimal line element

Case 2:

r and ϕ is constant

$$d\vec{r}_2 = r d\theta \hat{e}_\theta$$



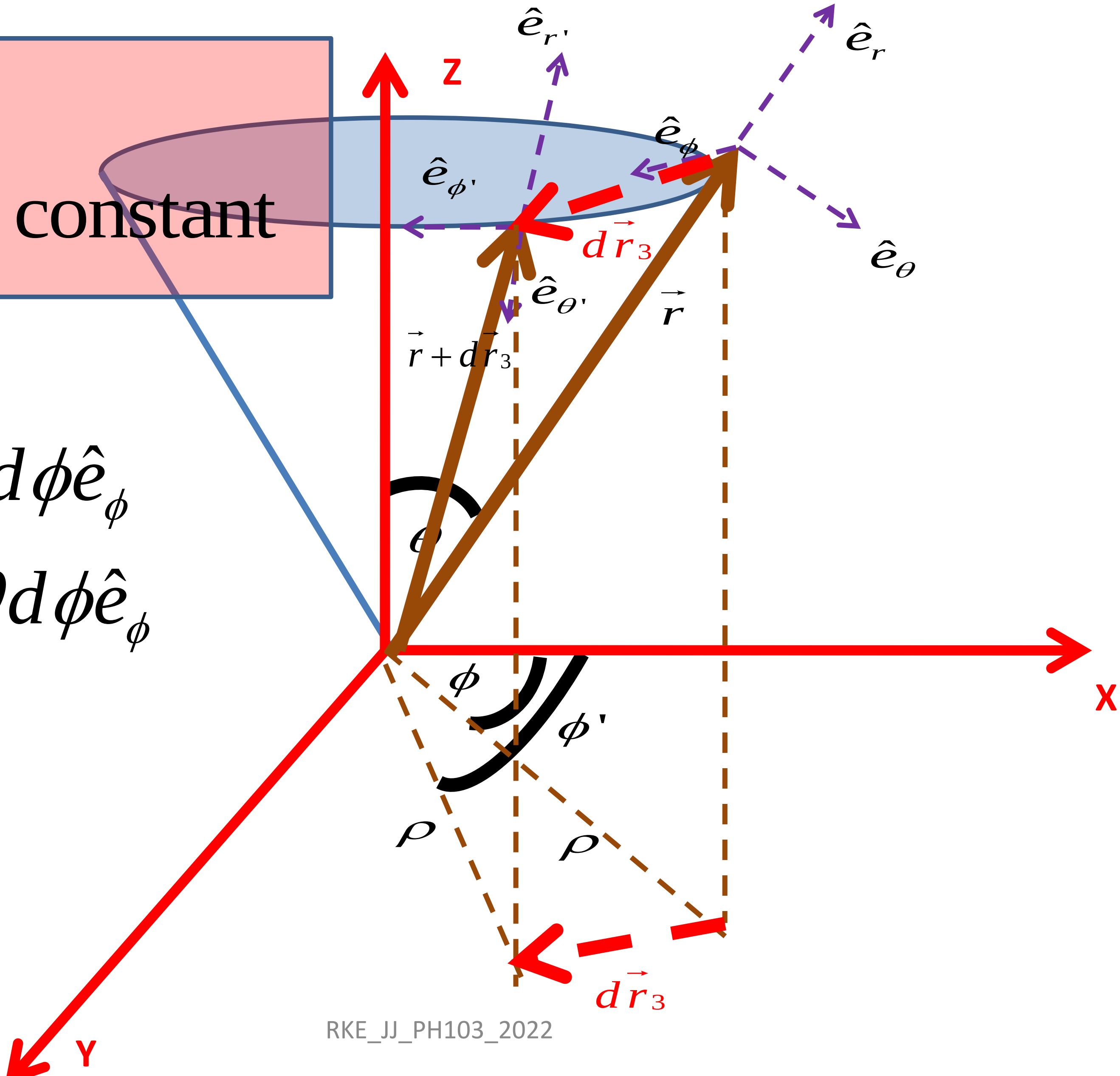
Infinitesimal line element

Case 3:

r and θ is constant

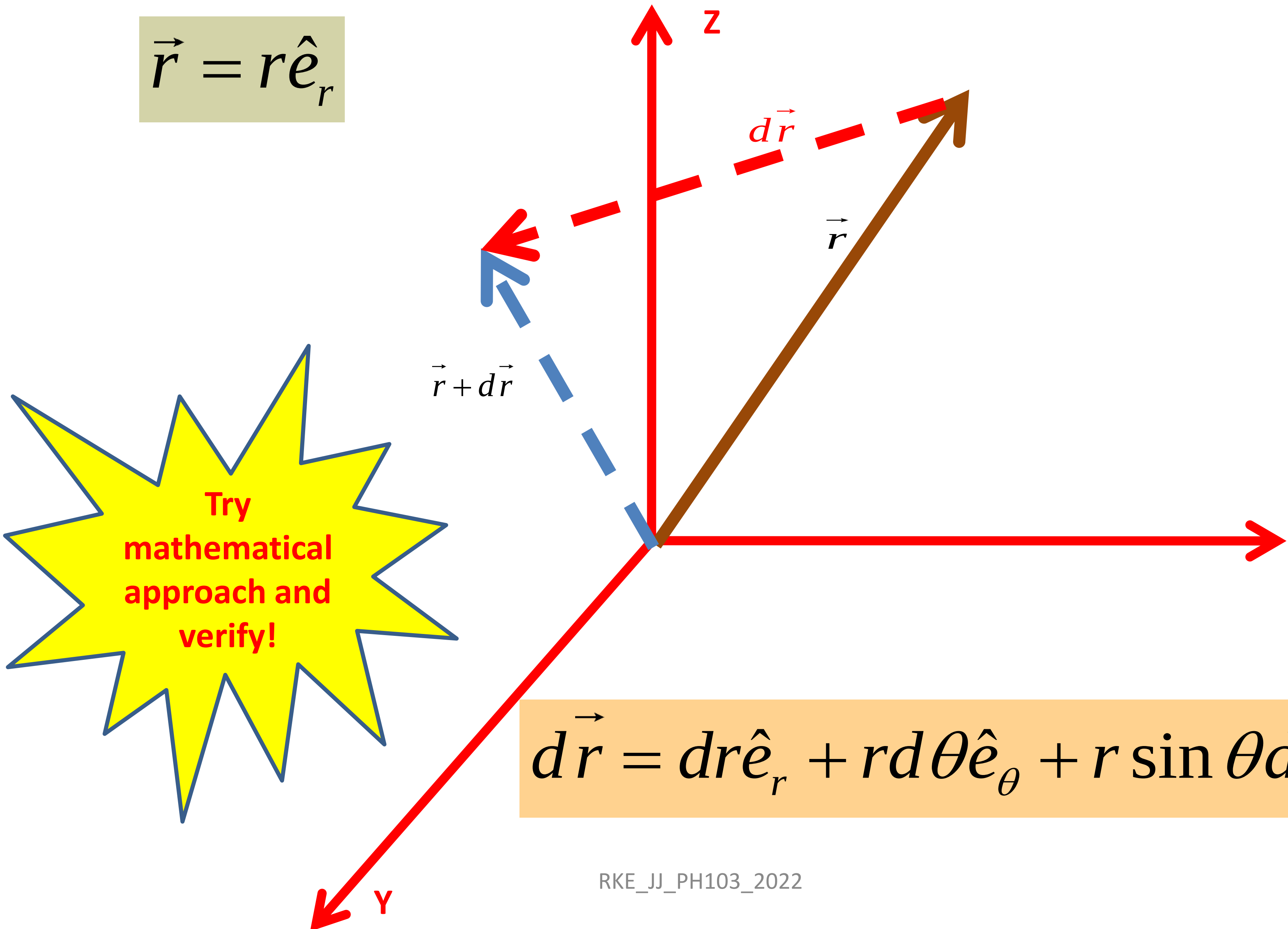
$$d\overrightarrow{r_3} = \rho d\phi \hat{e}_\phi$$

$$= r \sin\theta d\phi \hat{e}_\phi$$



Infinitesimal line element

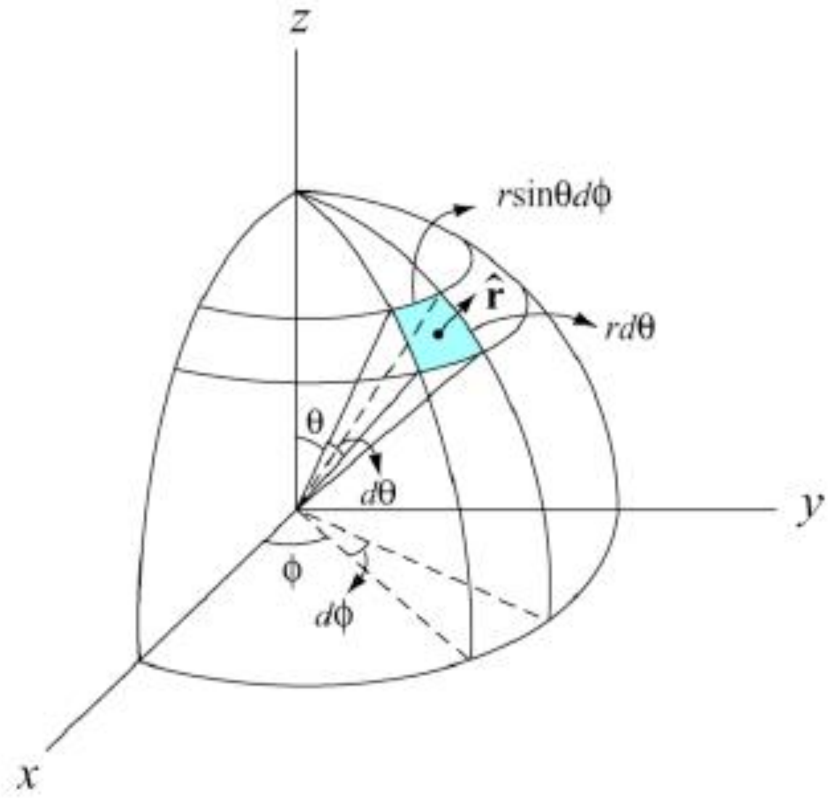
$$\vec{r} = r\hat{e}_r$$



$$d\vec{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$$

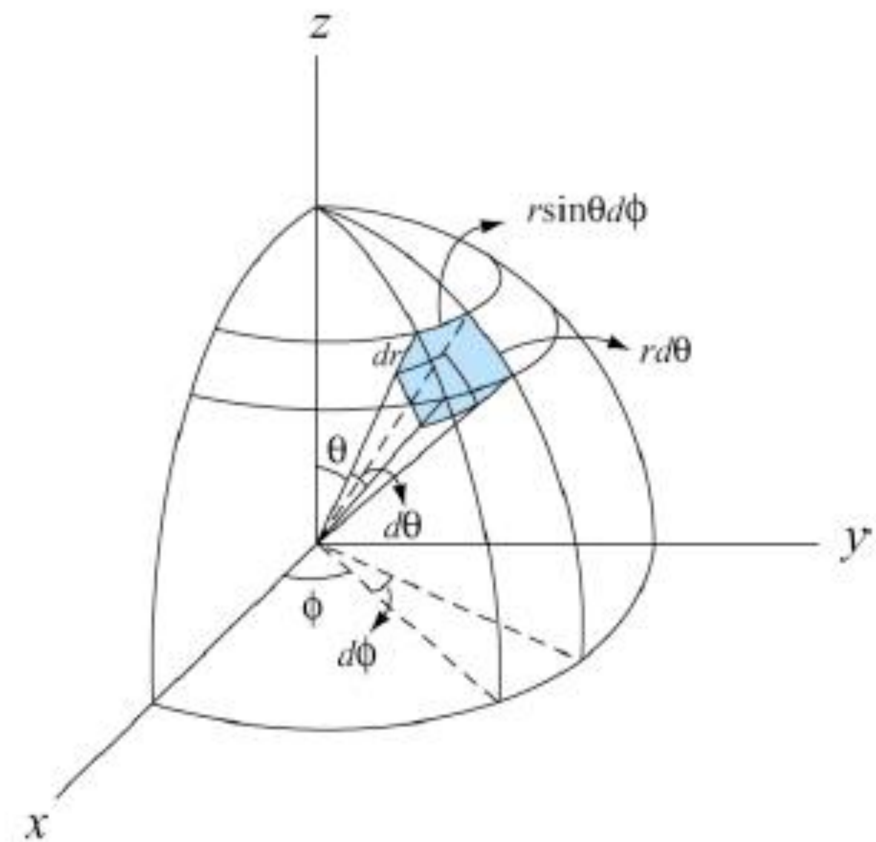
Infinitesimal area element

$$d\vec{A} = (rd\theta)(r\sin\theta d\phi)\hat{e}_r = r^2 \sin\theta d\theta d\phi \hat{e}_r$$



Infinitesimal volume element

$$dV = (dr)(rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta dr d\theta d\phi$$



Velocity and acceleration

$$\vec{r} = r\hat{e}_r$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\left[\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} + \frac{d\hat{e}_r}{d\phi}\frac{d\phi}{dt}\right]$$

$$\vec{V} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2$$

$$a_\theta = 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 + r\ddot{\theta}$$

$$a_\phi = 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta + r\sin\theta\,\ddot{\phi}$$

Cylindrical polar

$$(\rho, \phi, z)$$

$$\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z$$

$$d\vec{r} = d\rho\hat{e}_\rho + \rho d\phi\hat{e}_\phi + dz\hat{e}_z$$

$$\hat{e}_\rho = \hat{e}_x \cos(\phi) + \hat{e}_y \sin(\phi) + 0\hat{e}_z$$
$$\hat{e}_\phi = -\hat{e}_x \sin(\phi) + \hat{e}_y \cos(\phi) + 0\hat{e}_z$$
$$\hat{e}_z = \hat{e}_z$$

$$\vec{V} = \dot{\rho}\,\hat{e}_\rho + \rho\,\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{e}_\rho + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{e}_\phi + \ddot{z}\,\hat{e}_z$$

Spherical polar

$$(r, \theta, \phi)$$

$$\vec{r} = r\hat{e}_r$$

$$d\vec{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\phi\hat{e}_\phi$$

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

$$\vec{V} = \dot{r}\,\hat{e}_r + r\,\dot{\theta}\,\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2$$

$$a_\theta = 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2 + r\ddot{\theta}$$

$$a_\phi = 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta + r\sin\theta\,\ddot{\phi}$$