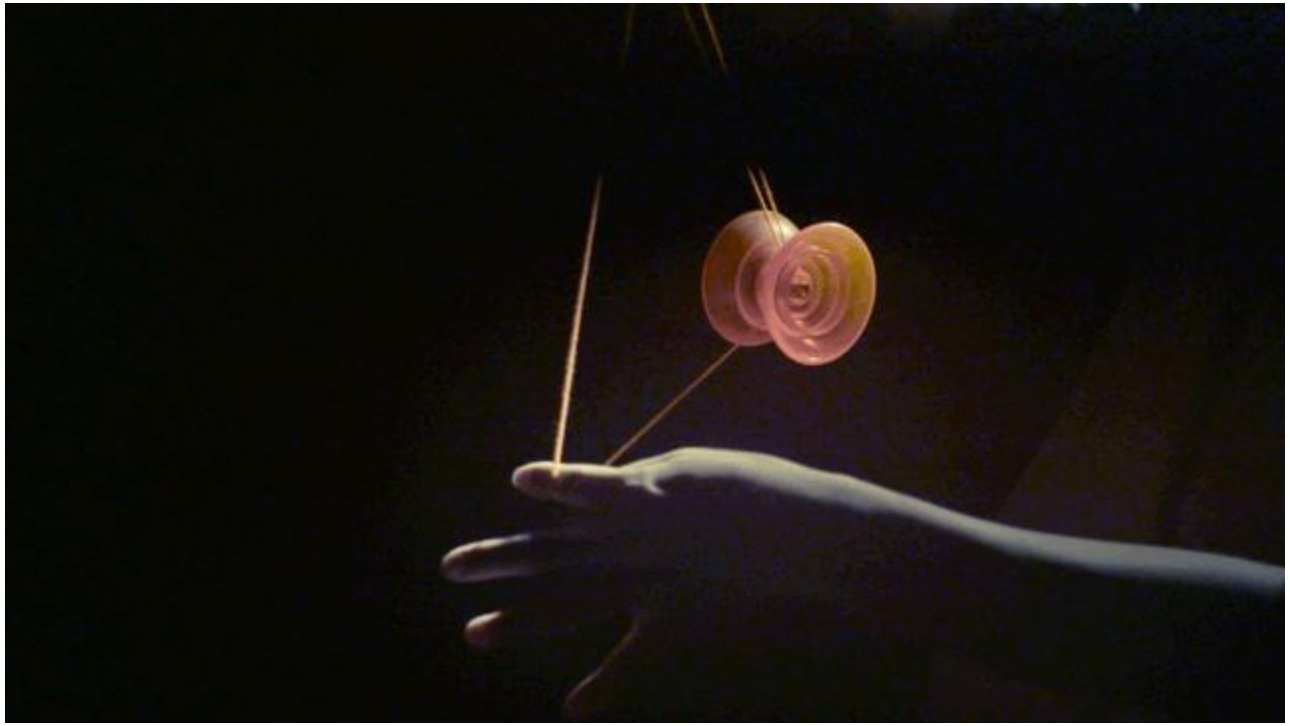
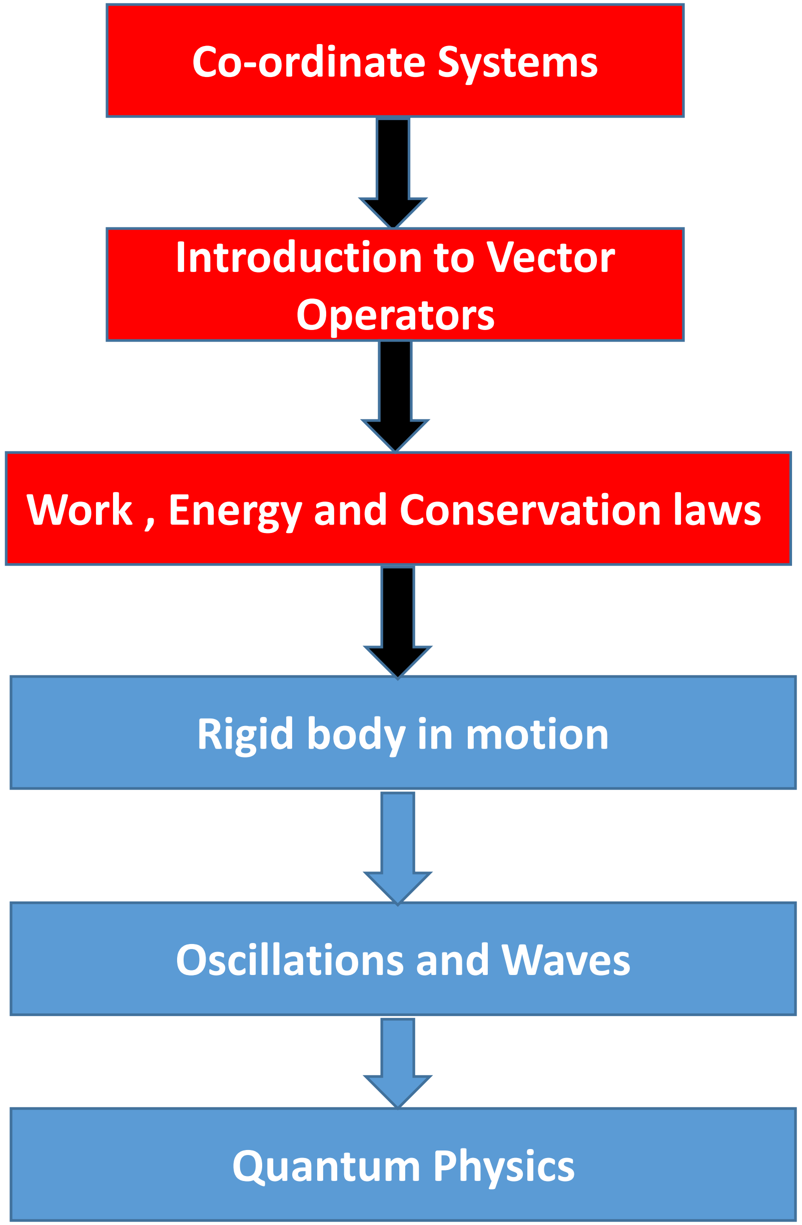
Chapter 4

RIGID BODY IN MOTION



Highlights of the course



Rotation Vs Translation

Translation

- Mass
- Linear momentum
- Force

Rotation

- Moment of Inertia
- Angular momentum
- Torque

Dynamics of Rotation [Recap]

$$\vec{\tau} = I\vec{\alpha}$$

$$\vec{F} = m\vec{a}$$

$\vec{\tau}$ = External Torque

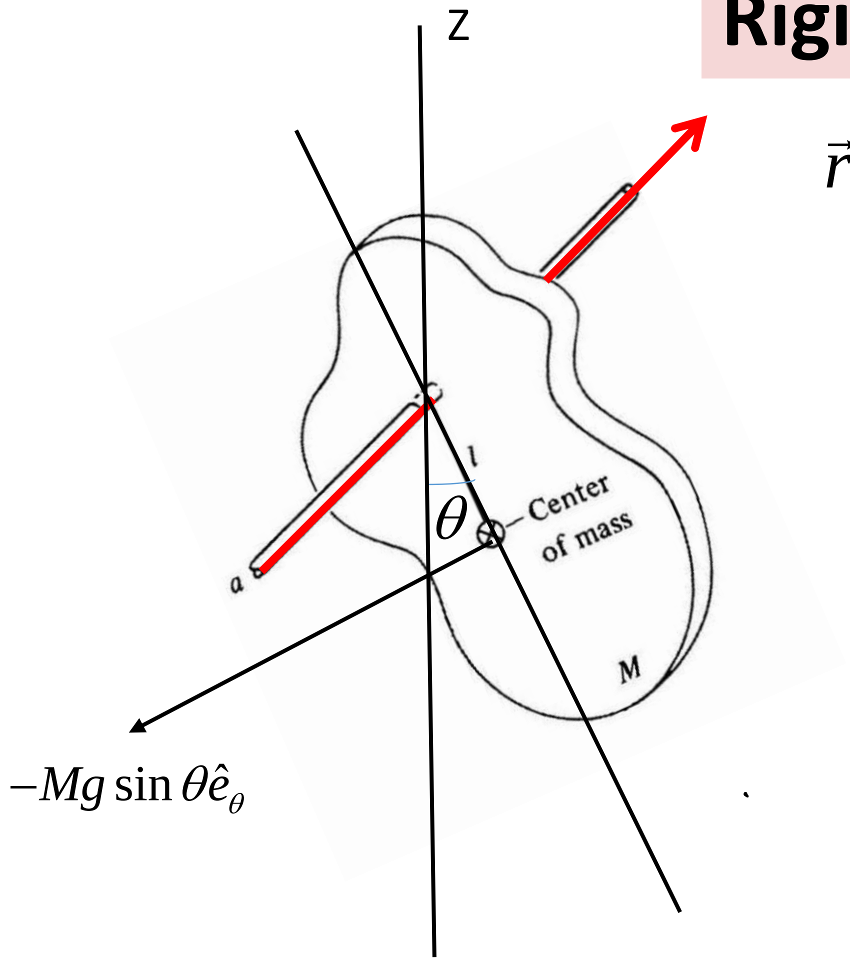
I=Moment of Inertia

α=Angular acceleration

Rotational Kinetic Energy

$$KE = \frac{1}{2} I\omega^2$$

Rigid body Pendulum [Recap]



$$\vec{r}\times\vec{F} = l\hat{e}_r \times\left(-Mg\sin\theta\hat{e}_\theta + Mg\cos\theta\hat{e}_r\right)$$

$$-lMg\sin\theta = I_a\ddot{\theta}$$

$$I_a\ddot{\theta} + lMg\theta = 0$$

$$\theta(t) = A\sin\omega t + B\cos\omega t$$

$$\omega = \sqrt{\frac{Ml\,g}{I_a}}$$

Work Energy Theorem

$$m\int_{x_a}^{x_b}\frac{dv}{dt}dx=\int_{x_a}^{x_b}F(x)dx$$

$$\frac{1}{2}mv_b{}^2-\frac{1}{2}mv_a{}^2=\int_{x_a}^{x_b}F(x)dx$$

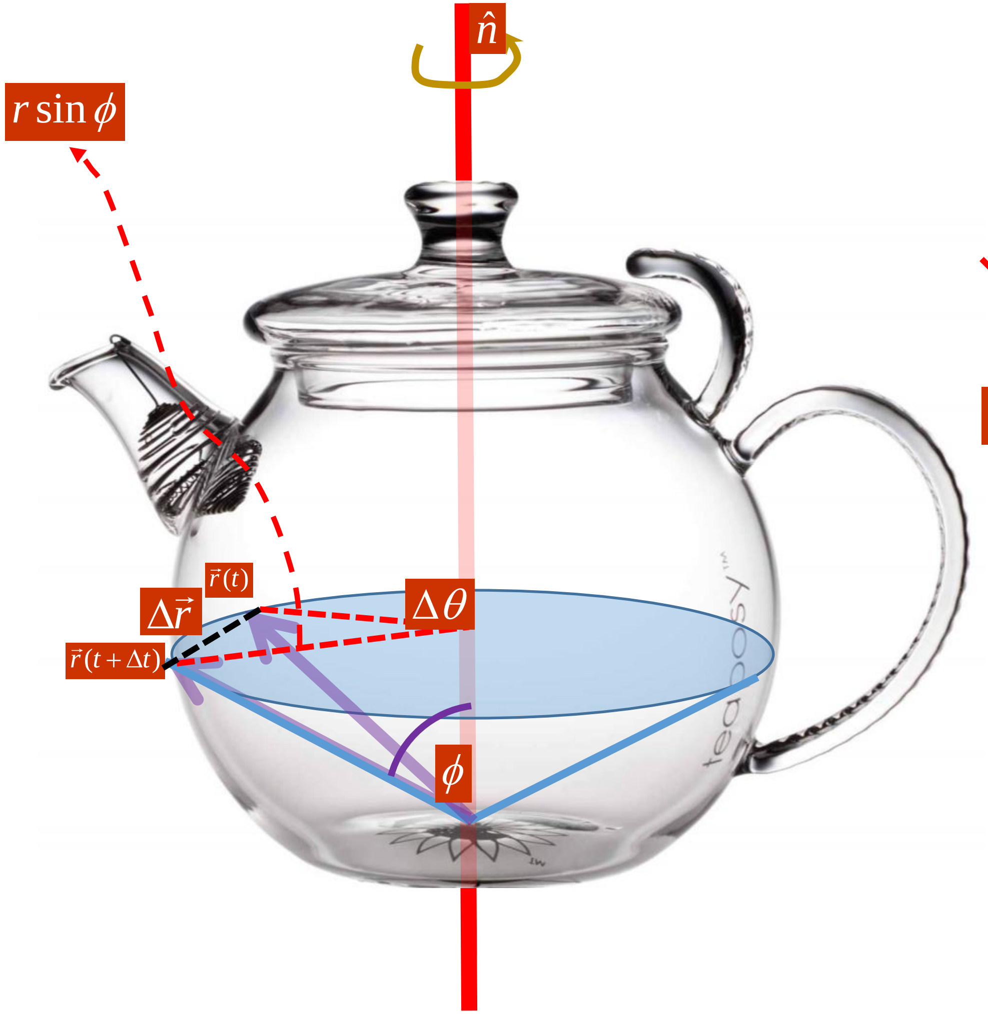
Work Energy Theorem for Rotational motion

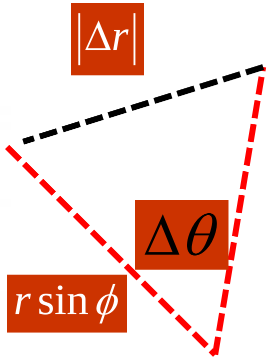
$$\frac{1}{2}I\omega_b{}^2-\frac{1}{2}I\omega_a{}^2=\int_{\theta_a}^{\theta_b}\tau d\theta$$

$$W_{ba}=K_b-K_a$$

Vector nature of angular velocity and angular momentum

Vector nature of angular velocity and angular momentum





For infinitesimal rotation,

$$|\Delta r| = r \sin\phi \, \Delta\theta$$

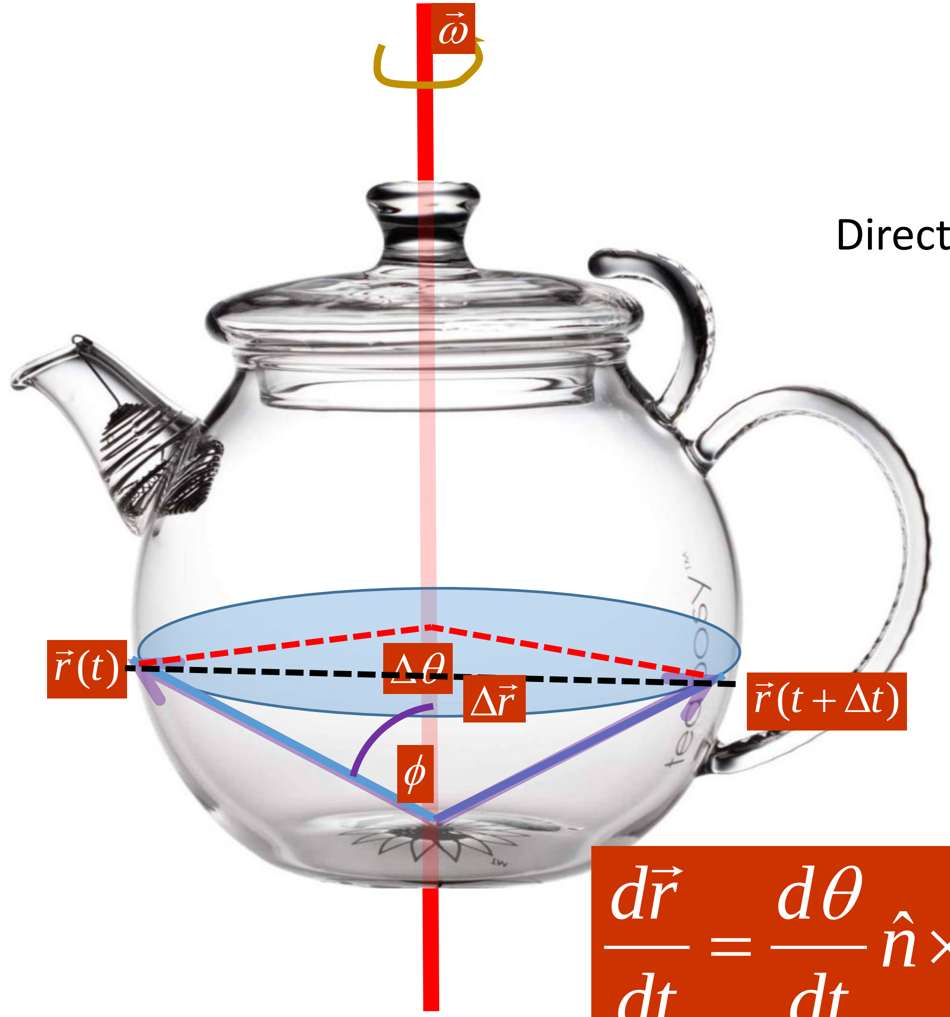
$$\left|\frac{dr}{dt}\right| = r \sin\phi \, \frac{d\theta}{dt}$$

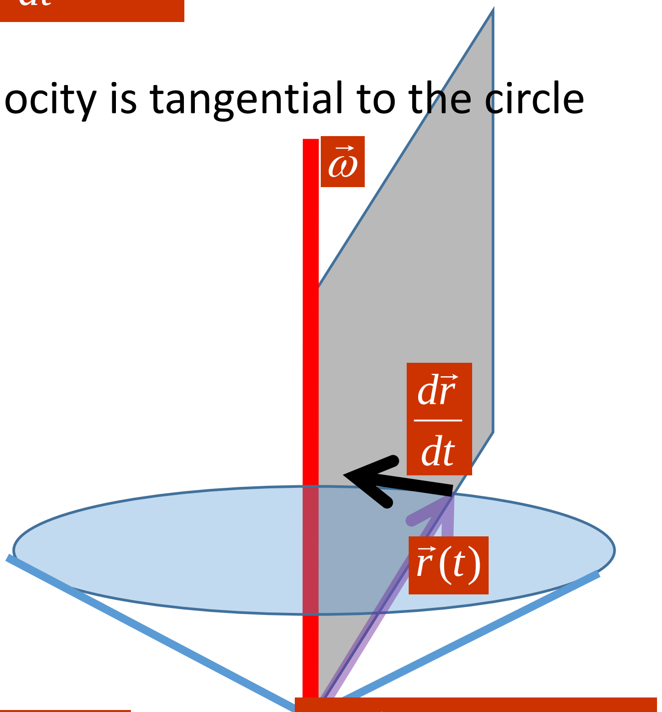
Vector nature **of angular velocity** and **angular momentum**

ω

$$\left|\frac{dr}{dt}\right| = \frac{d\theta}{dt} r \sin\phi$$

Direction of velocity is tangential to the circle

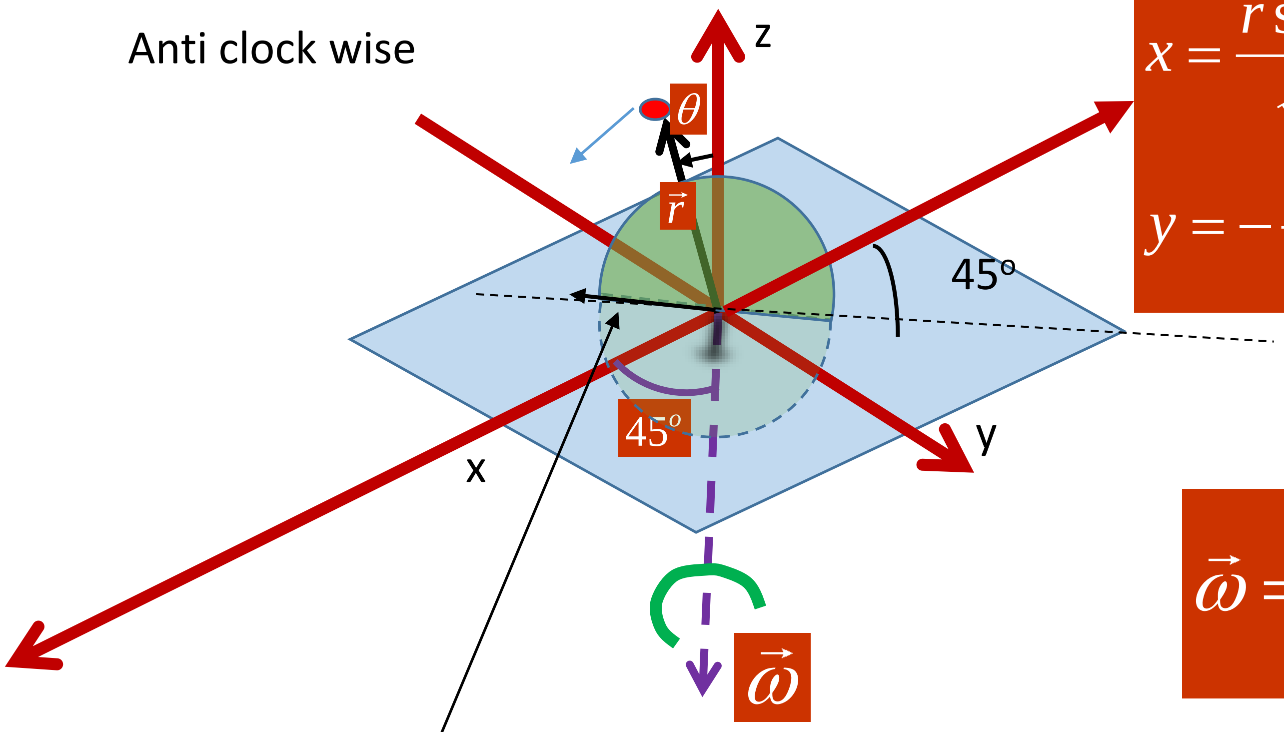




$$\frac{d\vec{r}}{dt} = \frac{d\theta}{dt}\hat{n}\times\vec{r} = \vec{\omega}\times\vec{r}$$

$$\frac{d\vec{A}}{dt} = \vec{\omega}\times\vec{A}$$

Find the velocity of a particle rotating in anti-clock wise direction in a vertical plane as shown in the diagram, the particle makes an angle 45 degrees with the –X–axis & +Y-axis and the axis of rotation makes 45 degree with X and Y axis (angular frequency ω make this angle)



Projection of $\vec{r} = r\sin\theta$

$$z = r\cos\theta$$

$$x = \frac{r\sin\theta}{\sqrt{2}}$$

$$y = -\frac{r\sin\theta}{\sqrt{2}}$$

$$\vec{r} = r\left(\frac{\sin\theta}{\sqrt{2}}\hat{e}_x - \frac{\sin\theta}{\sqrt{2}}\hat{e}_y + \cos\theta\hat{e}_z\right)$$

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r}$$

$$\vec{\omega} = \frac{\omega}{\sqrt{2}}\hat{e}_x + \frac{\omega}{\sqrt{2}}\hat{e}_y$$

$$\frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} = \omega r\left(\frac{\cos\theta}{\sqrt{2}}\hat{e}_x + \frac{\cos\theta}{\sqrt{2}}\hat{e}_y - \sin\theta\hat{e}_z\right)$$

Angular momentum of a rigid body

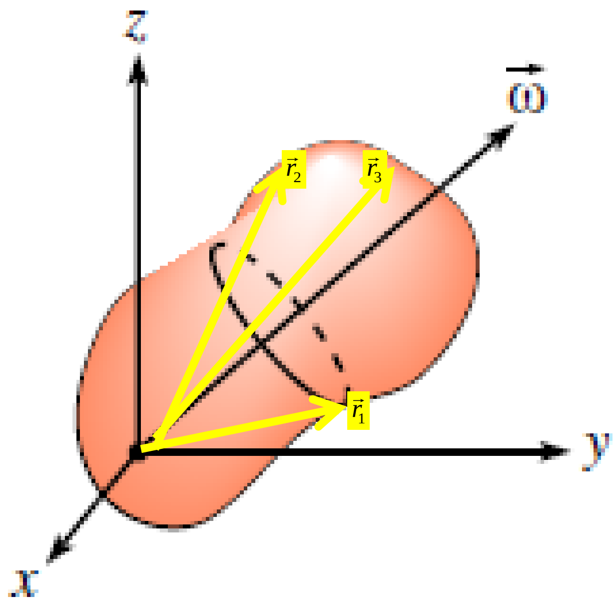
$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

$$\vec{L} = \vec{r} \times m\vec{v}$$

$$\vec{L} = [I][\vec{\omega}]$$

$$\vec{\tau} = [I][\vec{\alpha}]$$

Angular momentum of a rigid body



$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} = m\frac{d\vec{r}}{dt} = m(\vec{\omega} \times \vec{r})$$

$$\vec{L} = \sum_{j=1}^{N} \left[\vec{r}_j \times m_j \left(\vec{\omega} \times \vec{r}_j \right) \right]$$

$$\vec{\omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z$$

$$\vec{\omega} \times \vec{r}_j = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \omega_x & \omega_y & \omega_z \\ x_j & y_j & z_j \end{vmatrix}$$

$$\vec{r}_j \times m_j \left(\vec{\omega} \times \vec{r}_j \right)?$$

$$\vec{r}_j \times m_j\left(\vec{\omega}\times\vec{r}_j\right)?$$

$$\vec{\omega}\times\vec{r}_j = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \omega_x & \omega_y & \omega_z \\ x_j & y_j & z_j \end{vmatrix}$$

$$m_j\vec{\omega}\times\vec{r}_j = m_j(\omega_y z_j - y_j\omega_z)\hat{e}_x - m_j(\omega_x z_j - x_j\omega_z)\hat{e}_y + m_j(\omega_x y_j - x_j\omega_y)\hat{e}_z$$

$$\vec{r}_j \times m_j\left(\vec{\omega}\times\vec{r}_j\right) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ x_j & y_j & z_j \\ m_j(\omega_y z_j - y_j\omega_z) & -m_j(\omega_x z_j - x_j\omega_z) & m_j(\omega_x y_j - x_j\omega_y) \end{vmatrix}$$

$$L_x = m_j\left(y_j^2 + z_j^2\right)\omega_x - m_j x_j y_j \omega_y - m_j x_j z_j \omega_z$$

Angular momentum of a rigid body

$$\vec{L} = \sum_{j=1}^{N}\left[\vec{r}_j \times m_j\left(\vec{\omega}\times\vec{r}_j\right)\right]$$

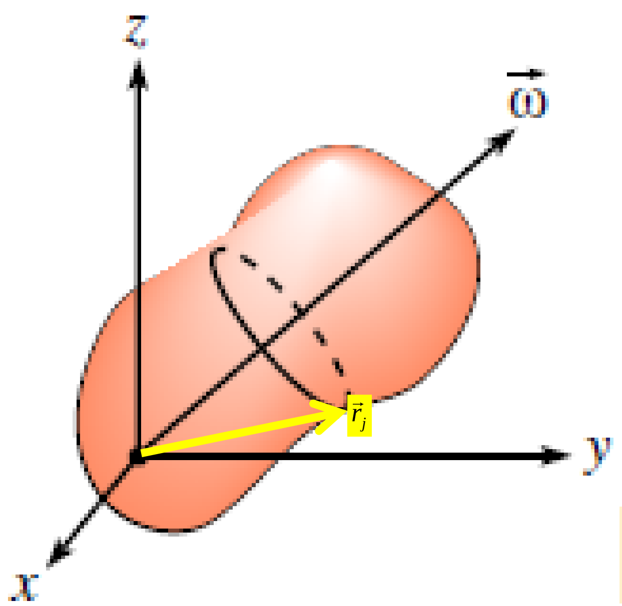
$$L_x = \sum_{j=1}^{N}\left[m_j\left(y_j^2+z_j^2\right)\omega_x - m_j x_j y_j \omega_y - m_j x_j z_j \omega_z\right]$$

Let us introduce moment of Inertias

$$I_{xx} = \sum m_j\left(y_j^2+z_j^2\right) \qquad I_{xy} = -\sum m_j x_j y_j \qquad I_{xz} = -\sum m_j x_j z_j$$

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

Angular momentum of a rigid body



$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

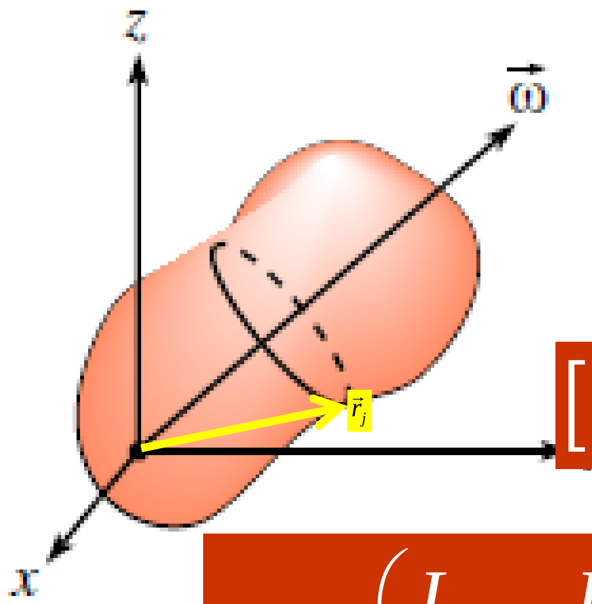
Different from what you learned!

Matrix Equation

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$[L] = [I][\omega]$$

Angular momentum of a rigid body



$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$[L] = [I][\omega]$ **Equivalent to** $\vec{p} = m\vec{v}$ **?**

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Moment of Inertia Matrix

Moment of Inertia Matrix

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{xx} = \sum m_j \left(y_j^2 + z_j^2 \right)$$

For a continuous medium,

$$I_{xx} = \int \left(y^2 + z^2 \right) dm$$

$$[I] = \begin{pmatrix} \int \left(y^2 + z^2 \right) dm & -\int xydm & -\int zxdm \\ -\int xydm & \int \left(z^2 + x^2 \right) dm & -\int yzdm \\ -\int zxdm & -\int yzdm & \int \left(x^2 + y^2 \right) dm \end{pmatrix}$$