Discrete Mathematics

Graphs-II

Arijit Mondal
Dept of CSE
arijit@iitp.ac.in

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

- **If every vertex of a graph G has degree at least 2 then G contains a cycle**

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

- **If every vertex of a graph G has degree at least 2 then G contains a cycle**
  - **Let P be a maximal path in G and let u be an endpoint of P,**

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

- **If every vertex of a graph G has degree at least 2 then G contains a cycle**
  - **Let P be a maximal path in G and let u be an endpoint of P,**
  - **Since P cannot be extended, every neighbor of u must already be a vertex in P**

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

- **If every vertex of a graph G has degree at least 2 then G contains a cycle**
  - **Let P be a maximal path in G and let u be an endpoint of P,**
  - **Since P cannot be extended, every neighbor of u must already be a vertex in P**
  - **Since u has degree at least 2, it has a neighbor v in $V(P)$ via an edge not in P**

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

- **If every vertex of a graph G has degree at least 2 then G contains a cycle**
  - **Let P be a maximal path in G and let u be an endpoint of P,**
  - **Since P cannot be extended, every neighbor of u must already be a vertex in P**
  - **Since u has degree at least 2, it has a neighbor v in $V(P)$ via an edge not in P**
  - **The edge uv completes a cycle with portion of P from v to u**

Eulerian trail

- **A graph is Eulerian if it has a closed trail containing all edges. An Eulerian circuit or Eulerian trail in a graph is a circuit or trail containing all edges.**

- **If every vertex of a graph G has degree at least 2 then G contains a cycle**
  - **Let P be a maximal path in G and let u be an endpoint of P,**
  - **Since P cannot be extended, every neighbor of u must already be a vertex in P**
  - **Since u has degree at least 2, it has a neighbor v in $V(P)$ via an edge not in P**
  - **The edge uv completes a cycle with portion of P from v to u**

- **A graph G is Eulerian if and only if it has at most one non-trivial component and its vertices all have even degree**

Vertex degree

- **The degree of a vertex in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v, except that each loop at v counts twice.**
- **The maximum degree is $\Delta(G)$, the minimum is $\delta(G)$ and G is regular if $\Delta(G) = \delta(G)$**

Degree-Sum formula

- **If G is graph, then $\sum_{v \in V(G)} d(v) = 2 \cdot e(G)$.**

Degree-Sum formula

- **If G is graph, then $\sum_{v \in V(G)} d(v) = 2 \cdot e(G)$.**
- **Summing the degrees counts each edge twice since each edge has two ends and contributes to the degree at each endpoint.**

Degree-Sum formula

- **If G is graph, then $\sum_{v \in V(G)} d(v) = 2 \cdot e(G)$.**
- **Summing the degrees counts each edge twice since each edge has two ends and contributes to the degree at each endpoint.**
- **In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence**
  $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$

Degree-Sum formula

- **If G is graph, then $\sum_{v \in V(G)} d(v) = 2 \cdot e(G)$.**
- **Summing the degrees counts each edge twice since each edge has two ends and contributes to the degree at each endpoint.**
- **In a graph G, the average vertex degree is $\frac{2e(G)}{n(G)}$, and hence**
  $\delta(G) \leq \frac{2e(G)}{n(G)} \leq \Delta(G)$
- **A k-regular graph with n vertices has $\frac{nk}{2}$ edges**

Example

- **For a simple graph G with vertices $v_1, \ldots, v_n$ and $n \geq 3$,**

$$e(G) = \frac{\sum_i e(G - v_i)}{n-2} \text{ and } d_G(v_j) = \frac{\sum_i e(G - v_i)}{n-2} - e(G - v_j)$$

Example

- **For a simple graph G with vertices $v_1, \ldots, v_n$ and $n \geq 3$,**

$$e(G) = \frac{\sum_i e(G - v_i)}{n-2} \text{ and } d_G(v_j) = \frac{\sum_i e(G - v_i)}{n-2} - e(G - v_j)$$

- **Summing the degrees counts each edge twice since each edge has two ends and contributes to the degree at each endpoint.**

Graphic sequences

- **The degree sequence of a graph is the list of vertex degrees, usually written in non-increasing order, as** $d_1 \geq d_2 \geq \ldots \geq d_n$

Graphic sequences

- **The degree sequence of a graph is the list of vertex degrees, usually written in non-increasing order, as $d_1 \geq d_2 \geq \ldots \geq d_n$**
- **The nonnegative integers $d_1, d_2, \ldots, d_n$ are vertex degrees of some graph if and only if $\sum_i d_i$ is even**
- **For $n > 1$, an integer list d of size n is graphic if and only if d' is graphic, where d' is obtained from d by deleting its largest element Δ and subtracting 1 from its Δ next largest elements. The only 1-element graphic sequence is $d_1 = 0$.**

Exercise

- **In a league with two divisions of 13 teams each, determine whether it is possible to schedule a season with each team playing nine games against teams within its division and,four games against teams in the other division.**

Exercise

- **Use complete graphs and counting arguments (not algebra!) to prove**
  - $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ **for** $0 \leq k \leq n$

Exercise

- **Use complete graphs and counting arguments (not algebra!) to prove**
  - $\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}$ **for** $0 \leq k \leq n$
  - **If** $\sum_i n_i = n$ **then** $\sum_i \binom{n_i}{2} \leq \binom{n}{2}$

Exercise

- **Prove that the number of simple even graphs with vertex set $[n]$ is $2^{\binom{n-1}{2}}$**

Exercise

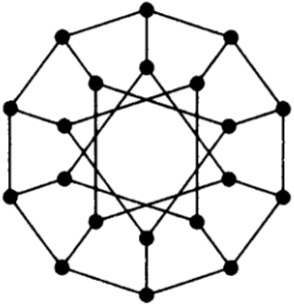
- **Which of the following are graphic sequences? Provide a construction or a proof of impossibility for each.**
  - $5, 5, 4, 3, 2, 2, 2, 1$
  - $5, 5, 4, 4, 2, 2, 1, 1$
  - $5, 5, 5, 3, 2, 2, 1, 1$
  - $5, 5, 5, 4, 2, 1, 1, 1$

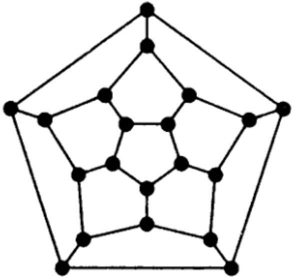
Exercise

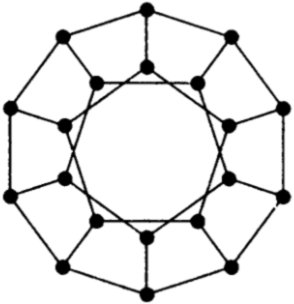
- **Count the cycles of length n in K_n and the cycles of length $2n$ in $K_{n,n}$.**

Exercise

- **Determine which pairs of graphs below are isomorphic.**







Exercise

- **Let G be the graph with vertex set $\{1, \ldots, 15\}$ in which *i* and *j* are adjacent if and only if their greatest common factor exceeds 1. Count the components of G and determine the maximum length of a path in G.**

Exercise

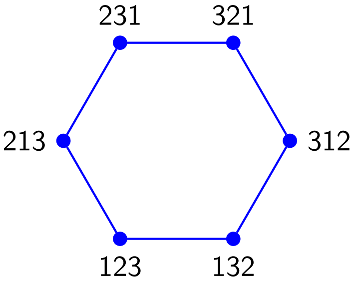
- **Determine the values of m and n such that $K_{m,n}$ is Eulerian.**

Exercise

- Let G be an n-vertex simple graph, where $n > 2$. Determine the maximum possible number of edges in G under each of the following conditions.
  - G has an independent set of size a.
  - G has exactly k components.
  - G is disconnected.

Exercise

- **Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$, with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by interchanging a pair of adjacent entries (G_3 shown below). Is G_n connected?**



Thank you!