Discrete Mathematics

Mathematical Induction

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Well-ordering

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  $\mathsf{Q}^+ = \{x \in \mathsf{Q} | x > 0\}$ **and** $\mathsf{R}^+ = \{x \in \mathsf{R} | x > 0\}$
- **Every non-empty subset of** Z^+ **contains a least / smallest element. This is not true for** Q^+**,** R^+

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  - Hence $F \in \emptyset$

Mathematical induction (contd.)

- **Let $S(n)$ denote an open mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer**

  a) If $S(n_0)$ is true for some $n_0 \in Z^+$; and

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- $[S(n_0) \wedge [\forall k \geq n_0 [S(k) \Rightarrow S(k+1)]]] \Rightarrow \forall n \geq n_0 S(n)$

Example: Harmonic series

- **Prove that,** $\sum_{k=1}^{n} H_k = (n+1)H_n - n$ **where** $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

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$$\begin{aligned}\sum_{j=1}^{k+1} H_j &= \sum_{j=1}^{k} H_j + H_{k+1} = [(k+1)H_k - k] + H_{k+1} \\ &= (k+1)\left[H_{k+1} - \frac{1}{k+1}\right] - k + H_{k+1} = (k+2)H_{k+1} - (k+1)\end{aligned}$$

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- **Incorrect proof !!**

Exercise-1

- **Prove that for every** $n \in \mathbb{Z}^+$ **where** $n \geq 14$, $S(n)$: n **can be written as a sum of** 3**'s and/or** 8**'s.**

Exercise-2

- **Consider the integer sequence** $a_0, a_1, a_2, \ldots$ **where** $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ **and** $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ **for all** $n \geq 3$**. Prove that** $a_n \leq 3^n$ **for all** $n \in \mathrm{Z}^+$

Exercise-3

- **Consider the following sequence of the numbers** $2021, 20821, 208821, 2088821, 20888821, \ldots$.
  Find a prime number that divides all these numbers. Prove it also.

Exercise-4

- **For a given $n \in \mathbb{Z}^+$, a composition of n is an ordered sum of positive-integer summands summing to n. Find the number of compositions for $1, 2, 3, 4$ and then generalize.**

Exercise-5

- $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. **Prove following**
  - $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$
  - $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
  - $F_{n-1} F_{n+1} = F_n^2 + (-1)^n$

Exercise-6

- Consider the following sequence a_n such that $a_0 = 9$ and $a_{n+1} = 3a_n^4 + 4a_n^3$ for $n > 0$. Show that a_{10} contains more than 1000 nines in decimal notation.

Thank you!