Discrete Mathematics

Proof Techniques

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Trivial proof

- **To prove, $p \rightarrow q$, we show q is true**

Trivial proof

- **To prove, $p \to q$, we show q is true**
- **Let $x \in \mathcal{R}$. If $x > 0$ then $x^2 + 5 > 0$**

Direct proof

- **To prove, $p \rightarrow q$, we show q is true under the assumption of p**

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Direct proof

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- The square of an odd integer is odd
- Every odd integer is equal to the difference between the squares of two integers
- If n is an odd integer, then $5n + 3$ is an even integer

Vacuous proof

- **To prove, $p \rightarrow q$, we show p is false**

Vacuous proof

- **To prove, $p \to q$, we show p is false**
- **Let $x \in \mathcal{R}$. If $x^2 + 1 < 0$ then $x^7 > 3$**

Exhaustive Proof and Proof by Cases

- To prove a conditional statement of the form
  - $(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$
  - $[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \Leftrightarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]$

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- **Prove that $(n+1)^3 \leq 3^n$ if n is a positive integer with $n \leq 4$**
- **Show that there are no solutions in integers x and y of $x^2 + 3y^2 = 8$**
- **Let $n \in \mathcal{Z}$. Then $n^2 + 3n + 5$ is an odd integer**

Existence proof

- **Many theorems are of the form** $\exists x \ p(x)$
- **Need to provide at least a single witness,** a **say, for the claim – constructive proof**
- **Instead providing a single witness, one can directly show** $\exists x \, p(x)$ **– non-constructive proof**

Constructive existence proof

- Show that there is a positive integer that can be written as sum of cubes of positive integers in two different ways.

Non-constructive existence proof

- **Show that there exist irrational numbers x and y such that x^y is rational**

Non-constructive existence proof

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- **Chomp is a game played by two players. In this game, cookies are laid out on a rectangular grid. The cookie in the top left position is poisoned. The two players take turns making moves; at each move, a player is required to eat a remaining cookie, together with all cookies to the right and/or below it. The loser is the player who has no choice but to eat the poisoned cookie. We ask whether one of the two players has winning strategy. That is, can one of the players always make moves that are guaranteed to lead to a win?**

Uniqueness proof

- To prove existence of a unique element with a particular property
  - Existence: We show that an element x with the desired property exists
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- Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Proof by contrapositive

- **To prove $p \rightarrow q$, we use an equivalent form of it – $\neg q \rightarrow \neg p$**
- **Assume $\neg q$ then prove $\neg p$**

Proof by contrapositive

- **To prove $p \to q$, we use an equivalent form of it – $\neg q \to \neg p$**
- **Assume $\neg q$ then prove $\neg p$**

- **Let $x \in \mathcal{Z}$. If $3x - 15$ is even, then x is odd**

Forward and backward reasoning

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- Suppose that two people play a game taking turns removing one, two, or three stones at a time from a pile that begins with 15 stones. The person who removes the last stone wins the game. Show that the first player can win the game no matter what the second player does.

Problems

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- Can you tile a board obtained by removing one of the four corners of the checkerboard
- Can you tile a board obtained by removing upper left and bottom right corner squares of the checkerboard

Law of invariance

- Suppose the positive integer n is odd. At first one writes the numbers $1, 2, \ldots, 2n$ on the blackboard. Then he picks any two numbers a, b, erases them, and writes, instead, $|a-b|$. Prove that an odd number will remain at the end.

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- **A circle is divided into six sectors. Then the numbers $1, 0, 1, 0, 0, 0$ are written into the sectors (counterclockwise, say). You may increase two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?**

Thank you!