Introduction to Deep Learning

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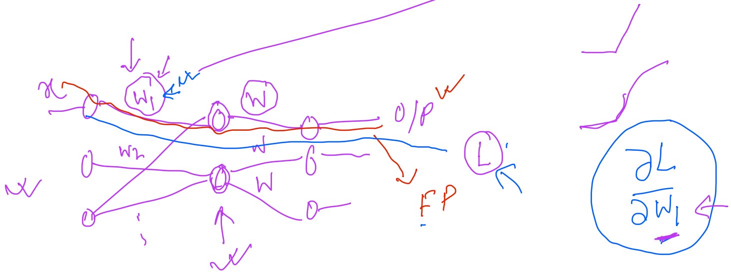
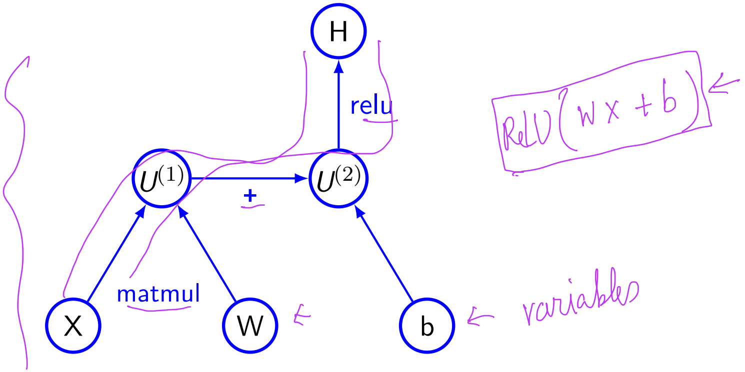
Backpropagation



Back propagation

- In a feedforward network, an input x is read and produces an output $\hat{y}$
  - This is forward propagation
- During training forward propagation continues until it produces cost $J(\boldsymbol{\theta})$
- Back-propagation algorithm allows the information to flow backward in the network to compute the gradient
- Computation of analytical expression for gradient is easy
- We need to find out gradient of the cost function with respect to the parameters ie. $\nabla_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$

Computational graph

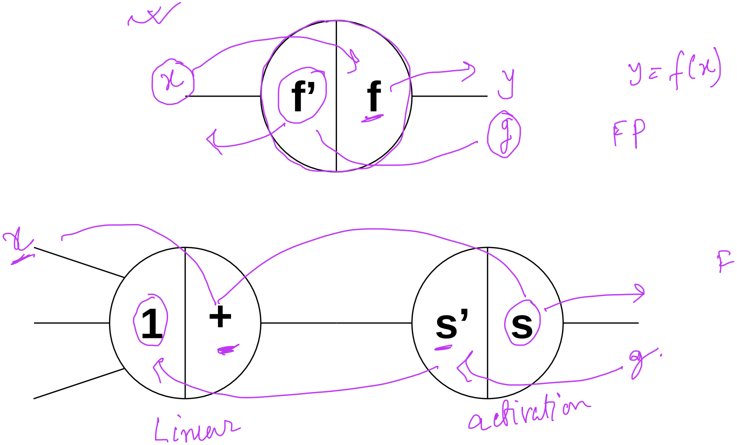


Chain rule of calculus

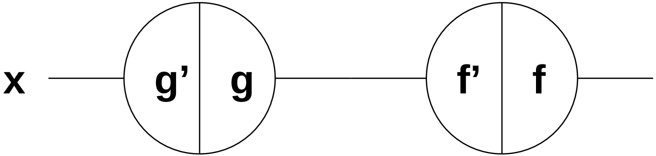
- Back-propagation algorithm heavily depends on it
- Let x be a real number and $y = g(x)$ and $z = f(g(x)) = f(y)$
- Chain rule says $\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$
- This can be generalized: Let $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{y} = g(\mathbf{x})$ and $z = f(\mathbf{y})$ then $\frac{\partial z}{\partial x_i} = \sum_j \frac{\partial z}{\partial y_j}\frac{\partial y_j}{\partial x_i}$
- In vector notation it will be where $\frac{\partial \mathbf{y}}{\partial \mathbf{x}}$ is the $n \times m$ Jacobian matrix of g

$$\nabla_{\mathbf{x}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)^T \nabla_{\mathbf{y}} z$$

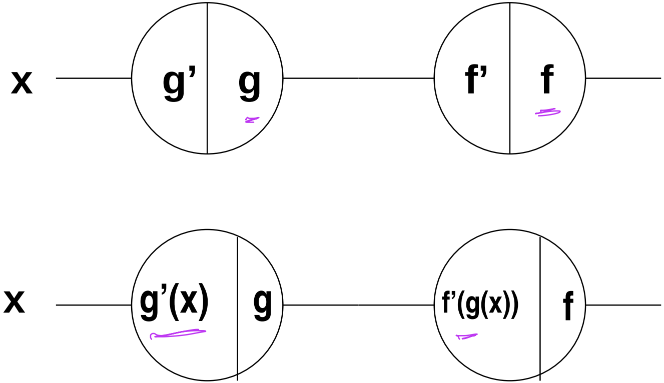
Back propagation



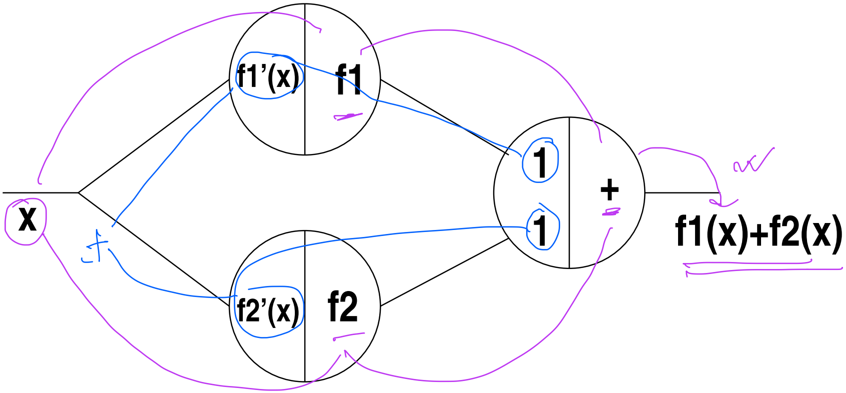
Back propagation



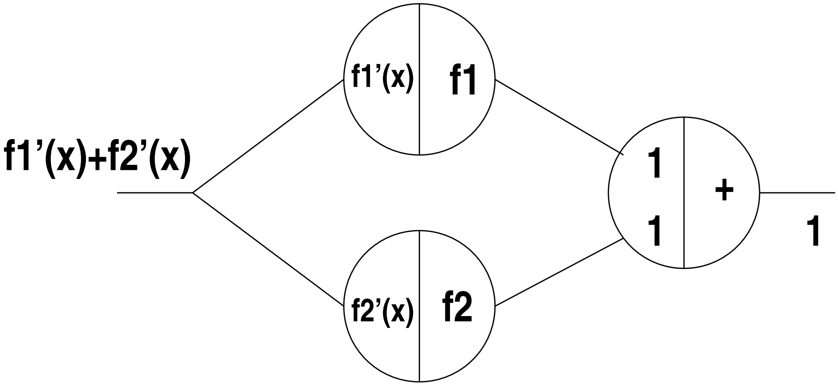
Back propagation



Back propagation



Back propagation

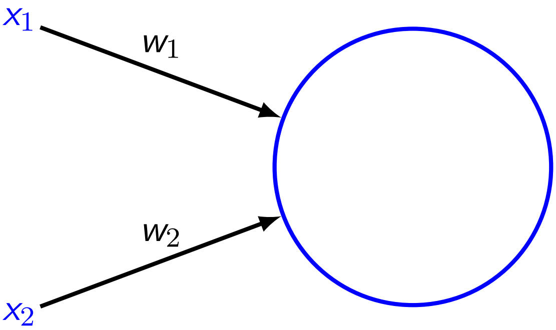


Backpropagation

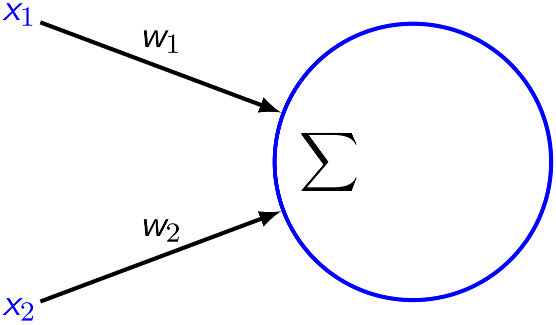
x_1

x_2

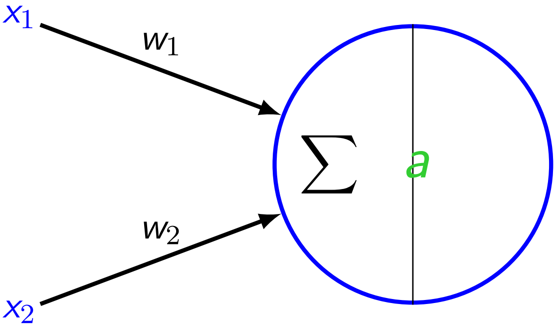
Backpropagation



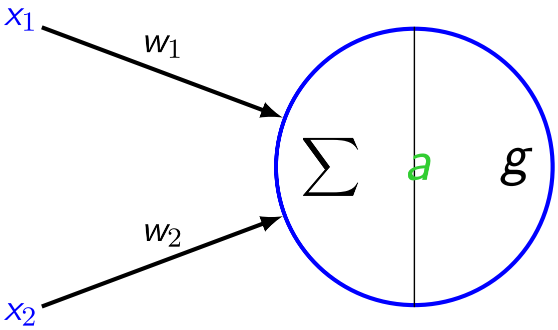
Backpropagation



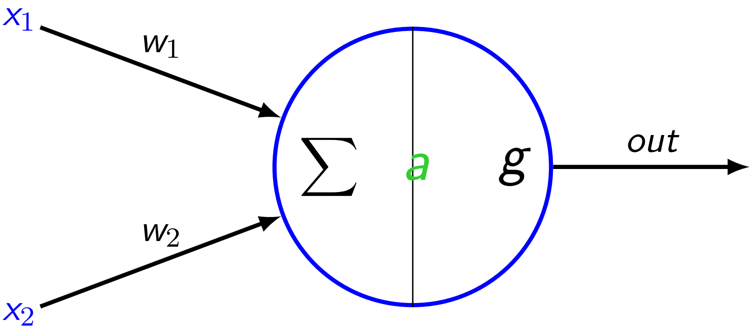
Backpropagation



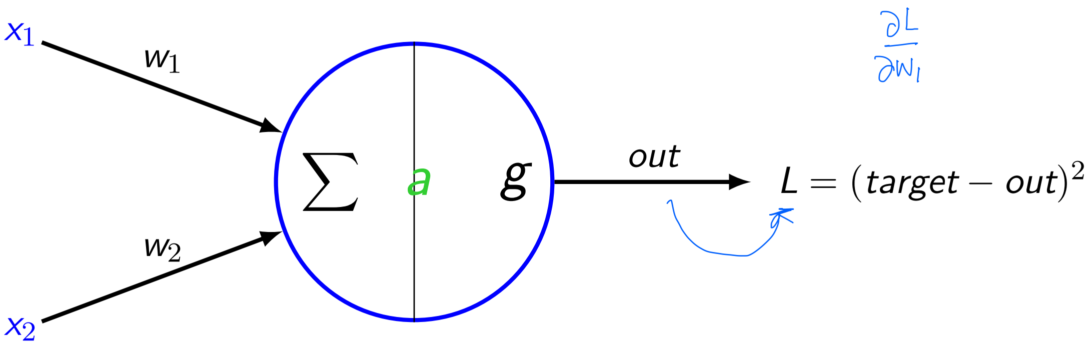
Backpropagation



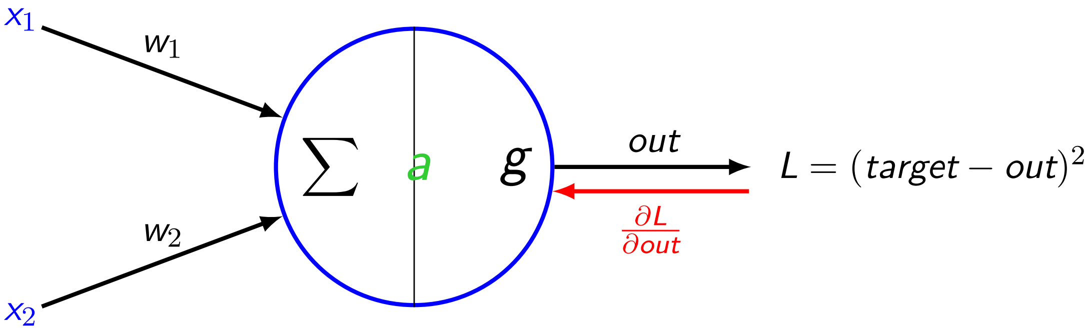
Backpropagation



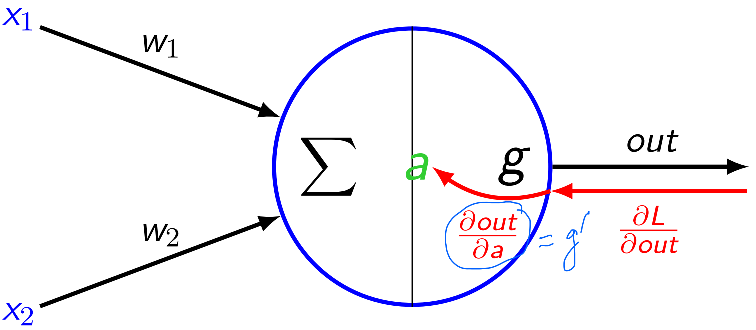
Backpropagation



Backpropagation

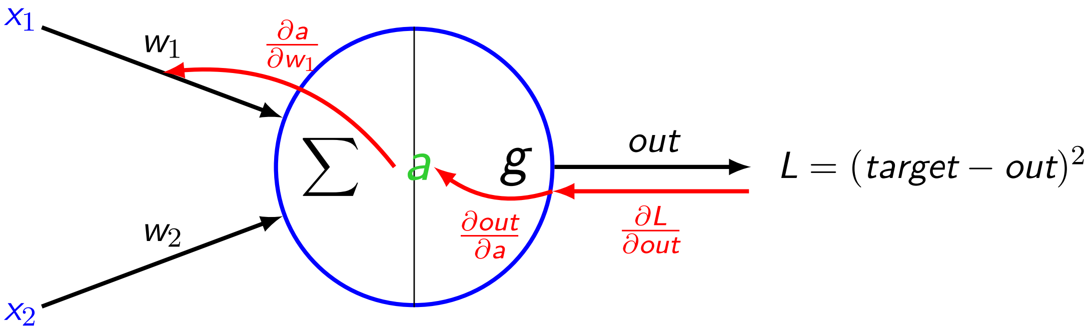


Backpropagation

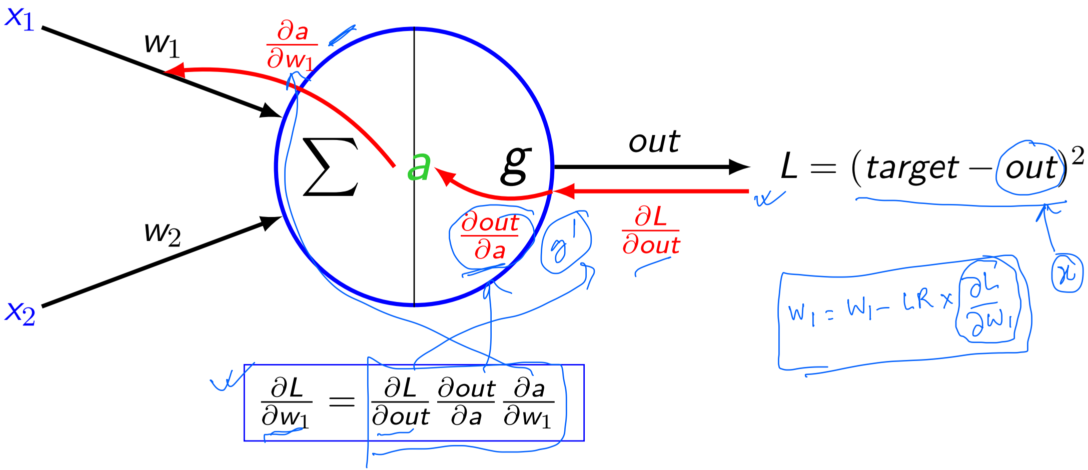


$$L = (target - out)^2$$

Backpropagation



Backpropagation

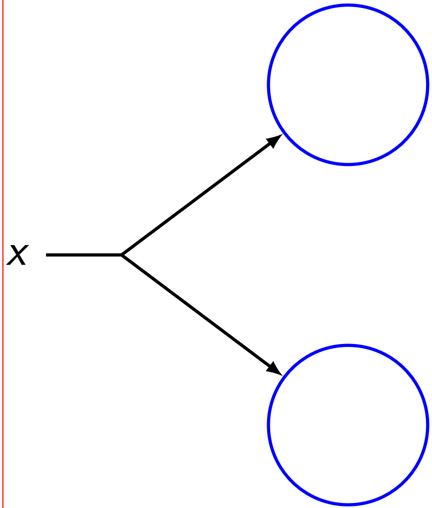


$$\frac{\partial L}{\partial w_1} = \frac{\partial L}{\partial out}\frac{\partial out}{\partial a}\frac{\partial a}{\partial w_1}$$

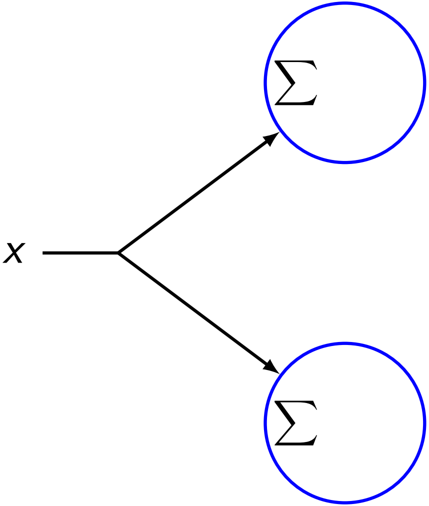
Backpropagation

x ——

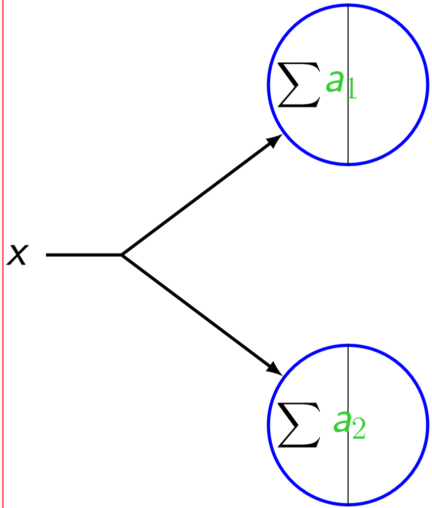
Backpropagation



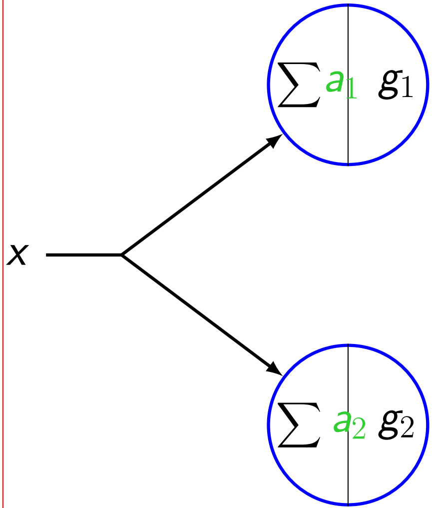
Backpropagation



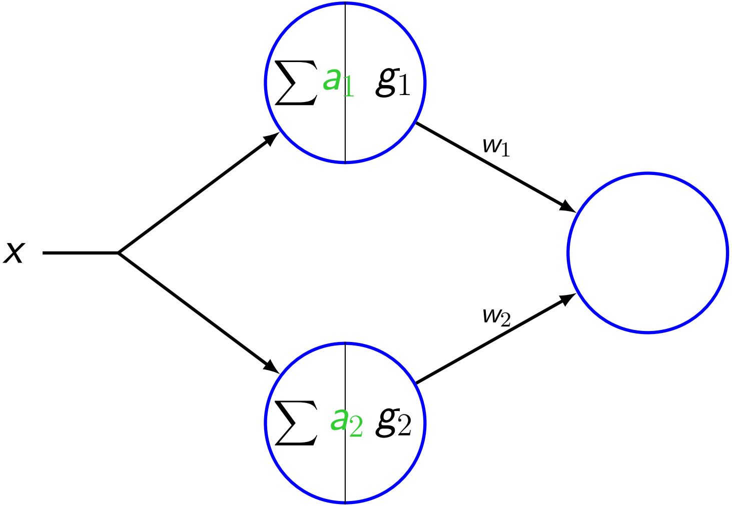
Backpropagation



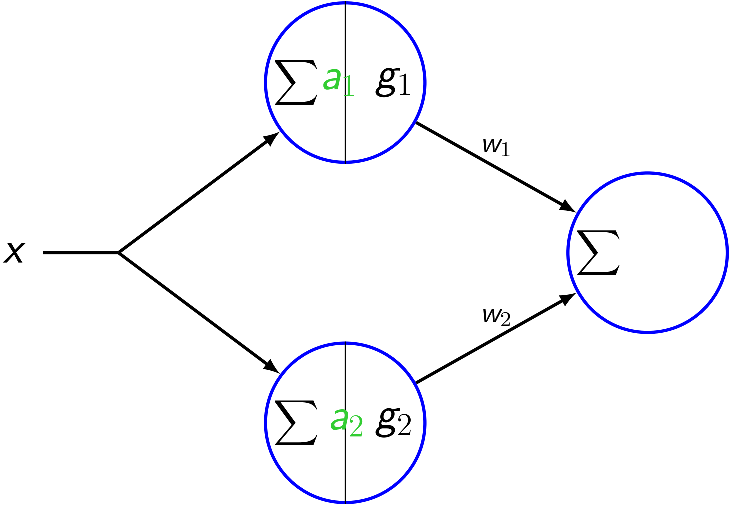
Backpropagation



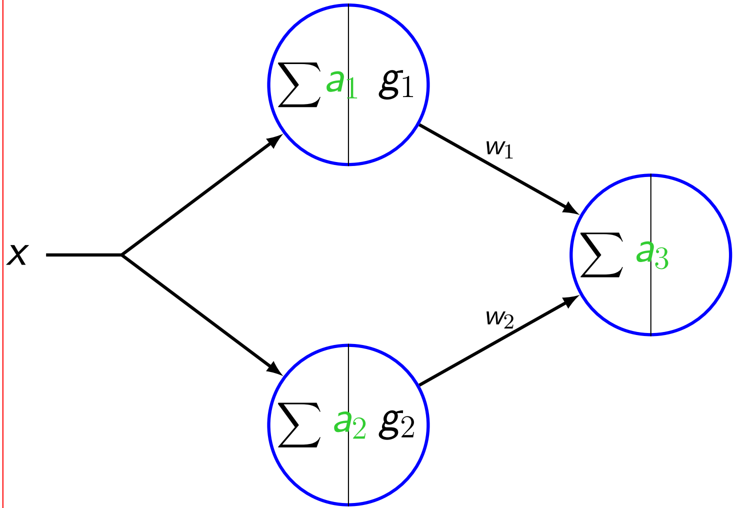
Backpropagation



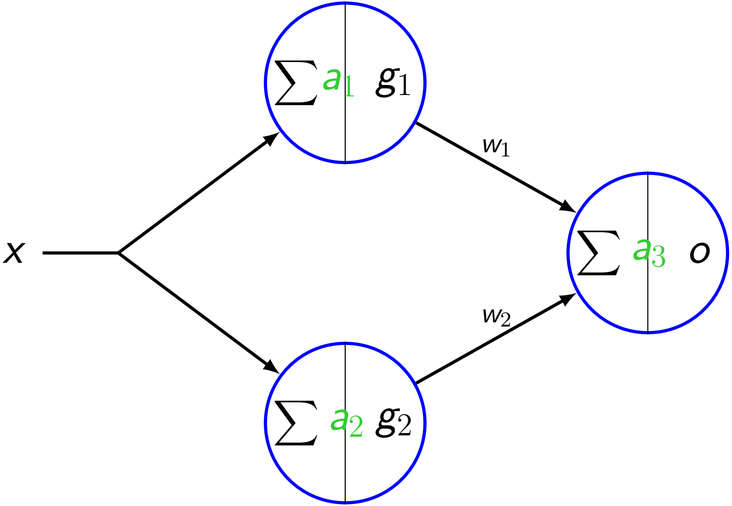
Backpropagation



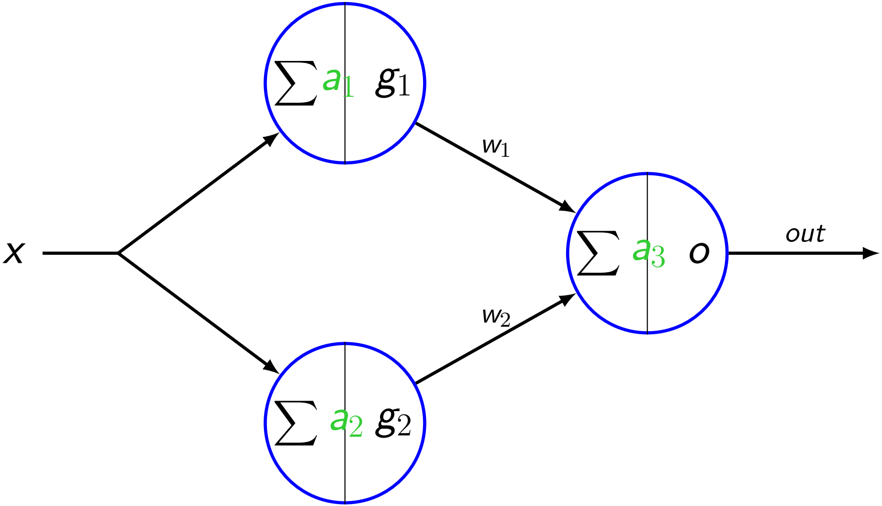
Backpropagation



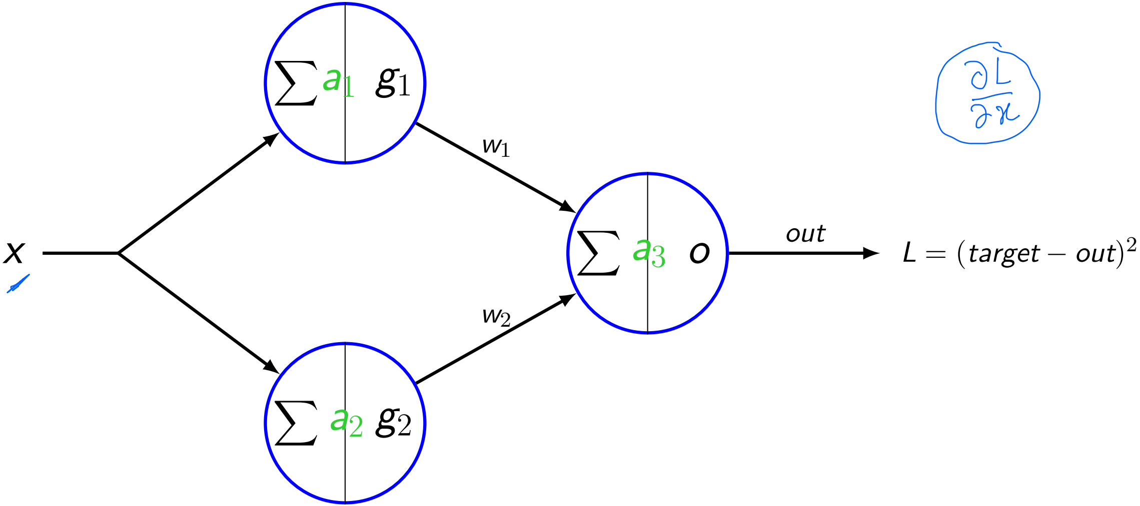
Backpropagation



Backpropagation



Backpropagation



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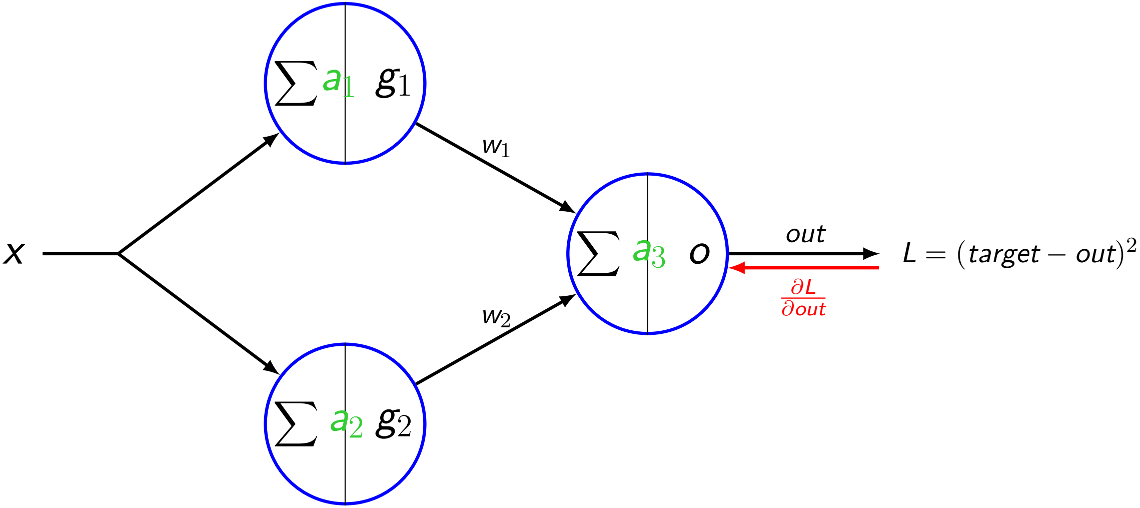
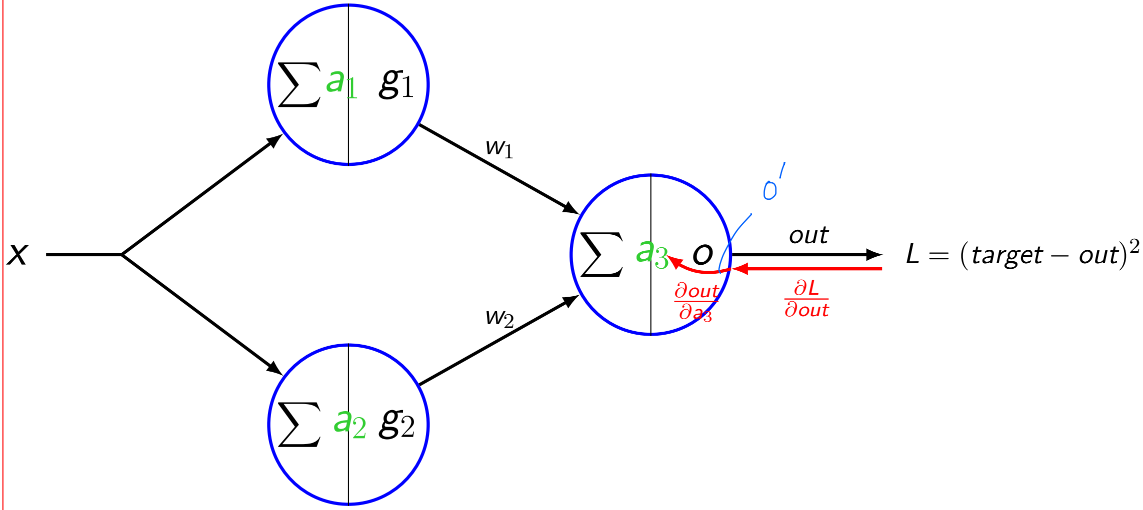
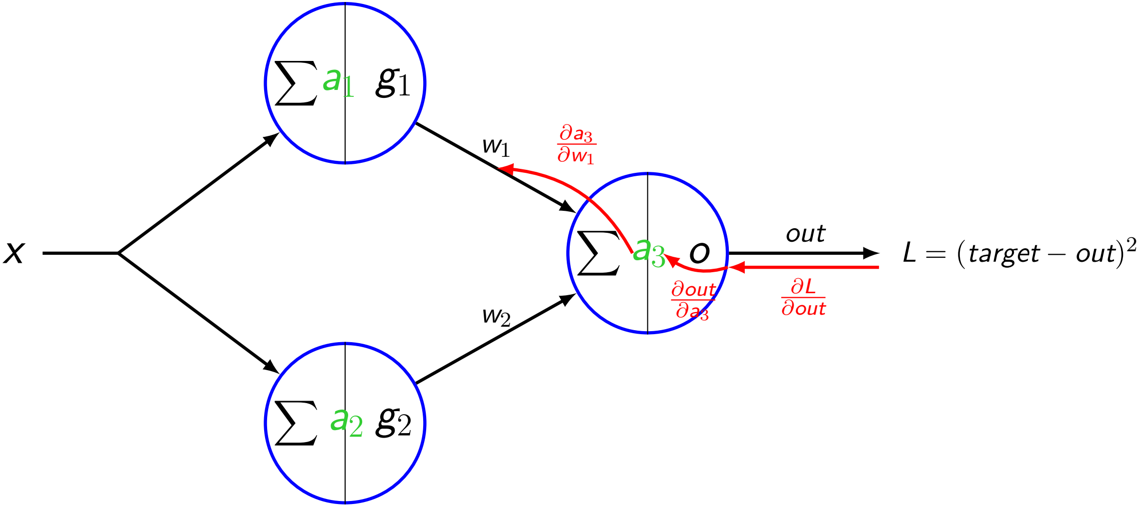


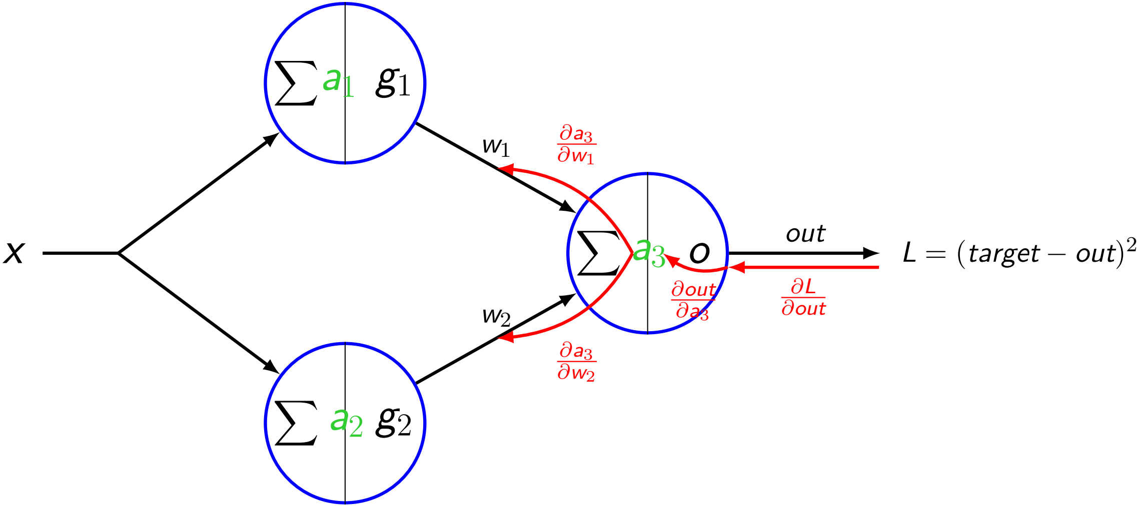
Backpropagation



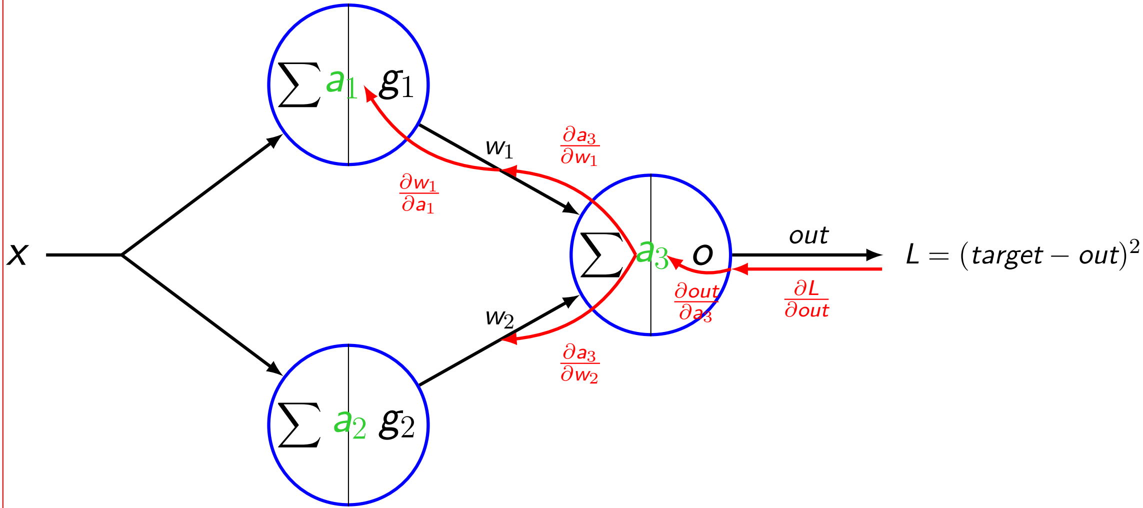
Backpropagation



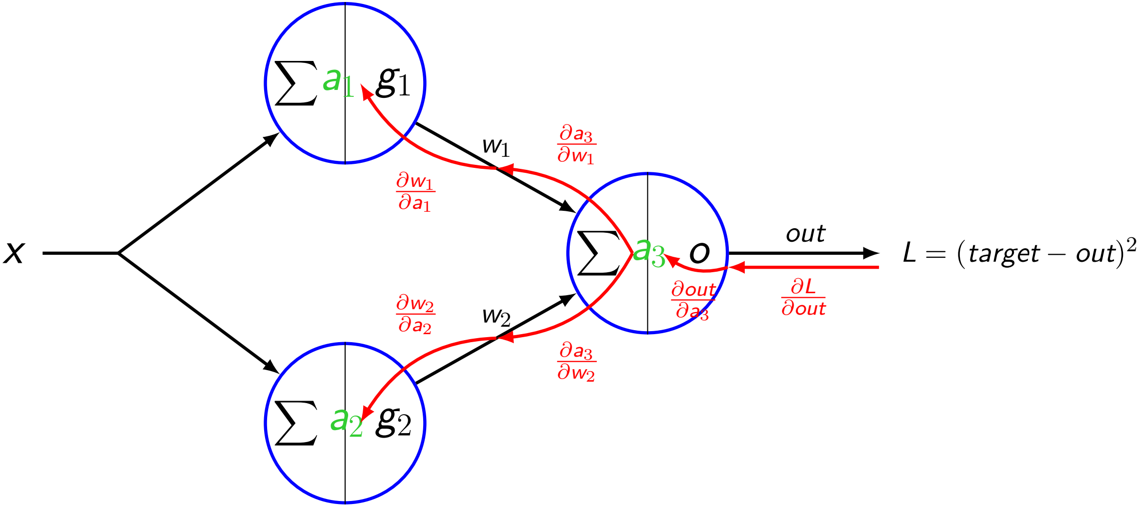
Backpropagation



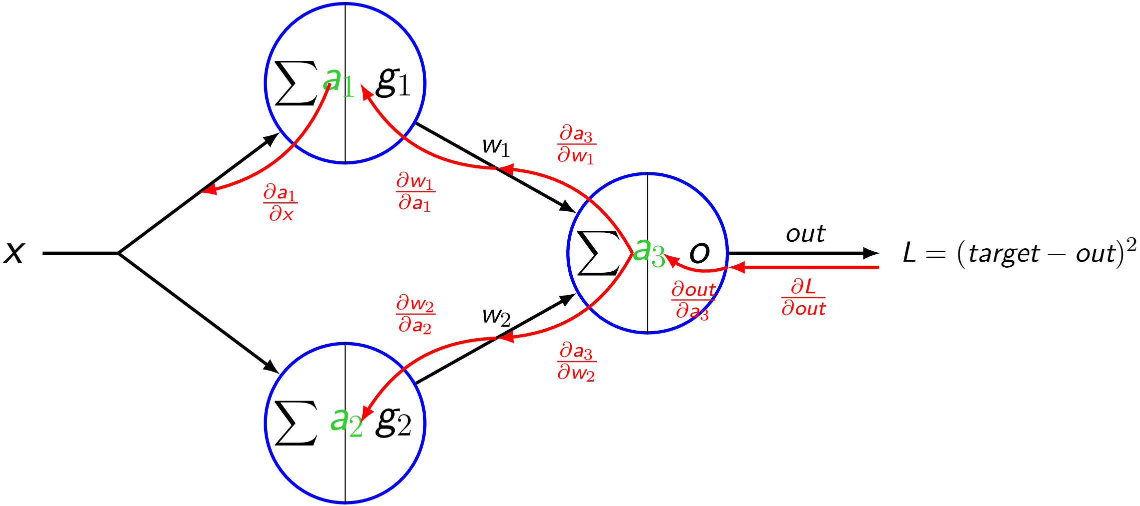
Backpropagation



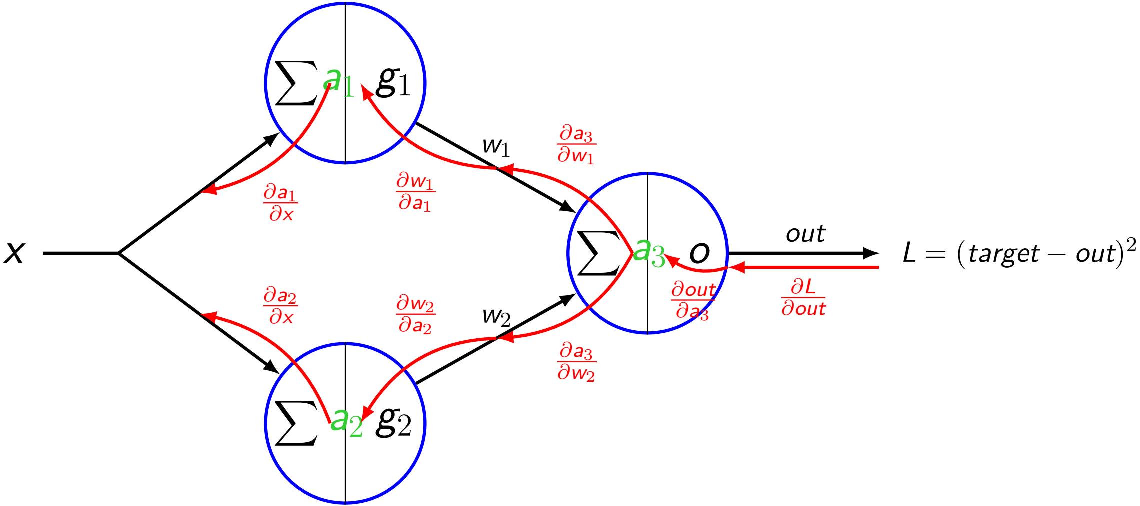
Backpropagation



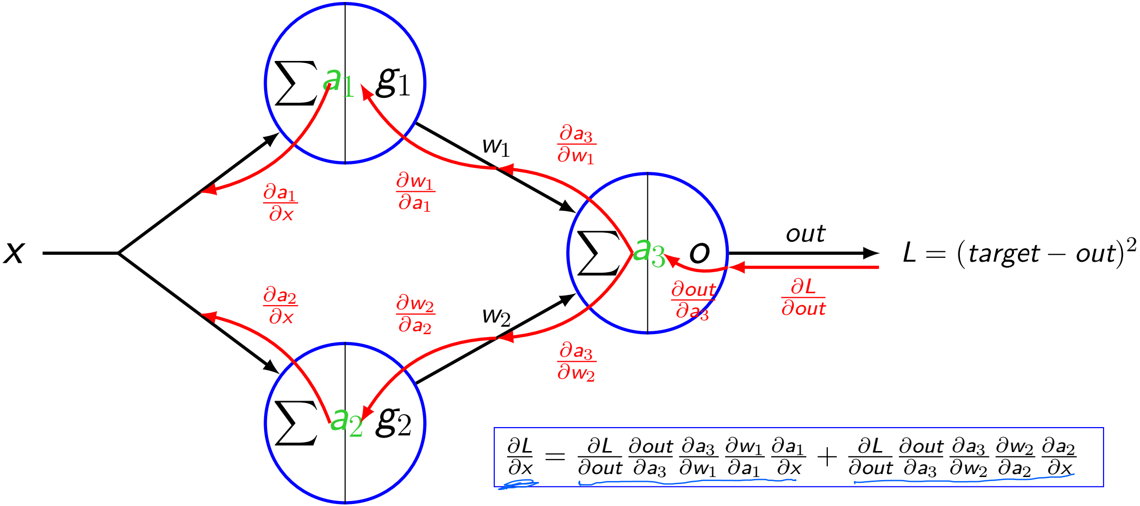
Backpropagation



Backpropagation

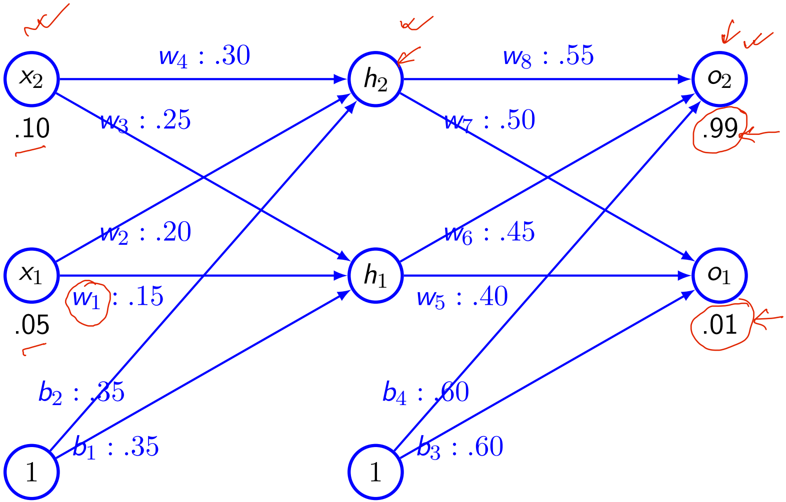


Backpropagation



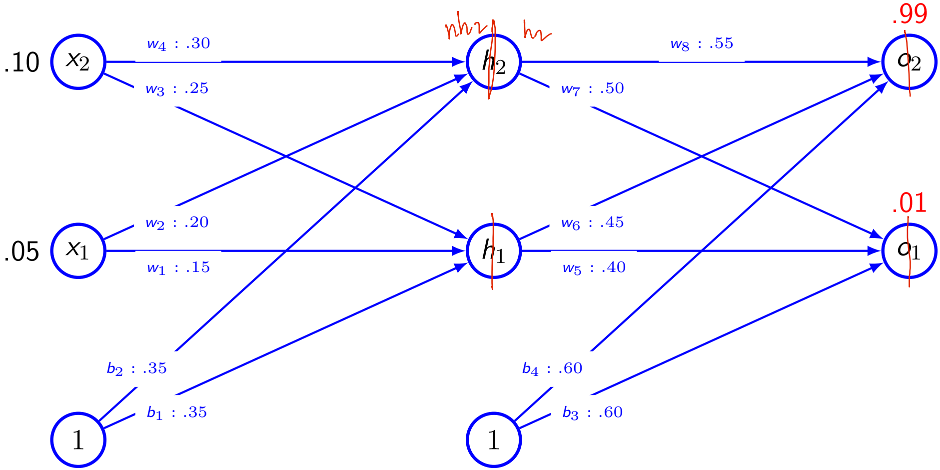
$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial out}\frac{\partial out}{\partial a_3}\frac{\partial a_3}{\partial w_1}\frac{\partial w_1}{\partial a_1}\frac{\partial a_1}{\partial x} + \frac{\partial L}{\partial out}\frac{\partial out}{\partial a_3}\frac{\partial a_3}{\partial w_2}\frac{\partial w_2}{\partial a_2}\frac{\partial a_2}{\partial x}$$

Example

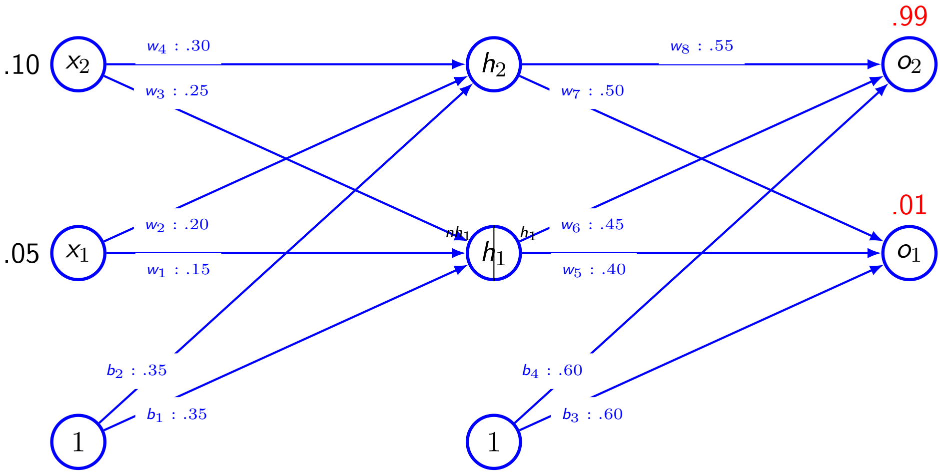


Hidden and output layer have sigmoid activation function. Loss function - MSE.

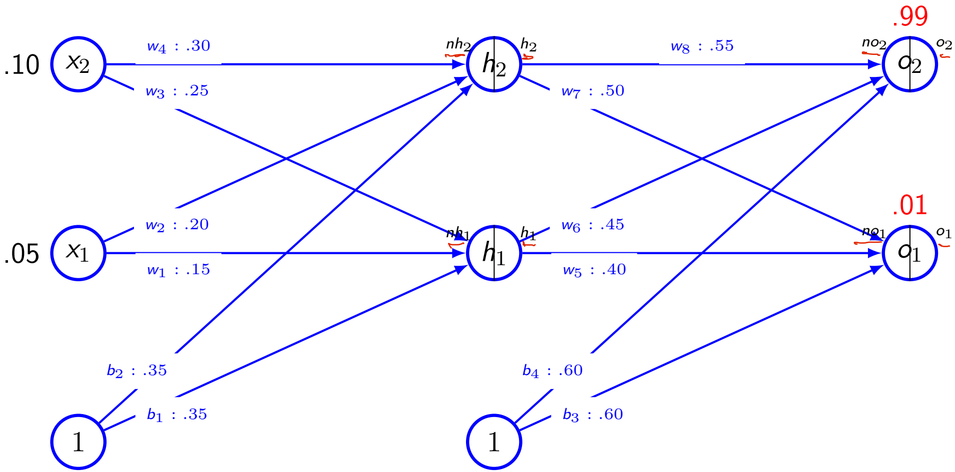
Example


.10
x_2
.05
x_1
1
w_4 : .30
w_3 : .25
w_2 : .20
w_1 : .15
b_2 : .35
b_1 : .35
h_2
h_1
1
w_8 : .55
w_7 : .50
w_6 : .45
w_5 : .40
b_4 : .60
b_3 : .60
o_2
o_1
.99
.01

Example

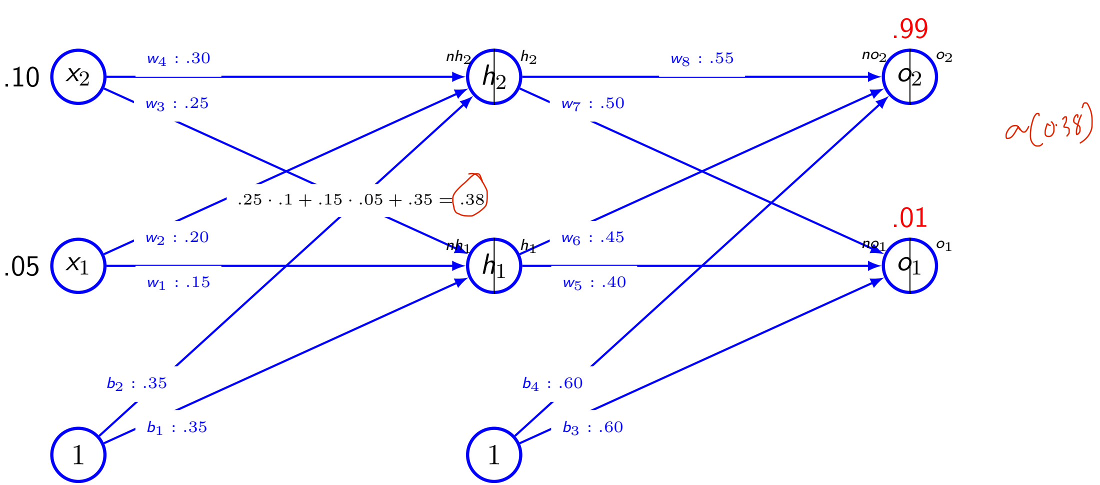

.99
.01
.10
.05
x_2
x_1
1
h_2
h_1
1
o_2
o_1
w_4 : .30
w_3 : .25
w_2 : .20
w_1 : .15
b_2 : .35
b_1 : .35
w_8 : .55
w_7 : .50
w_6 : .45
w_5 : .40
b_4 : .60
b_3 : .60

Example



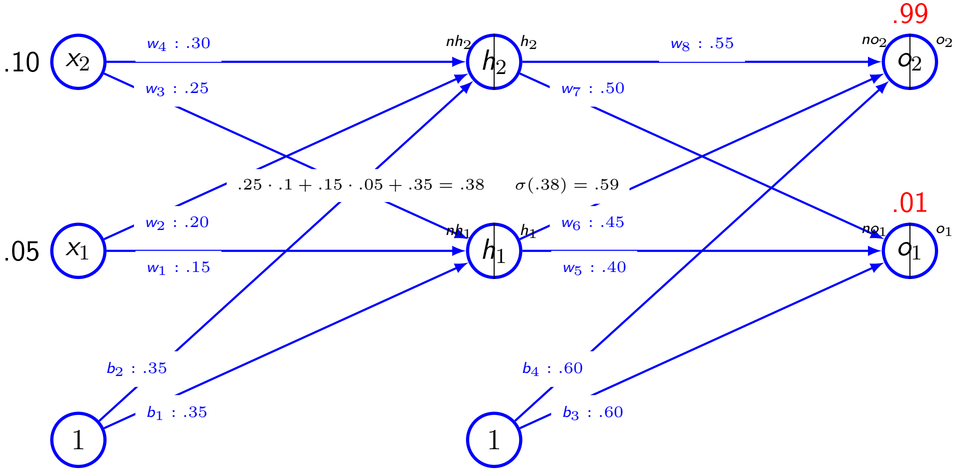
.99
.01
.10
.05
x_2
x_1
h_2
h_1
o_2
o_1
1
1
nh_2
h_2
nh_1
h_1
no_2
o_2
no_1
o_1
w_4 : .30
w_3 : .25
w_2 : .20
w_1 : .15
w_8 : .55
w_7 : .50
w_6 : .45
w_5 : .40
b_2 : .35
b_1 : .35
b_4 : .60
b_3 : .60

Example



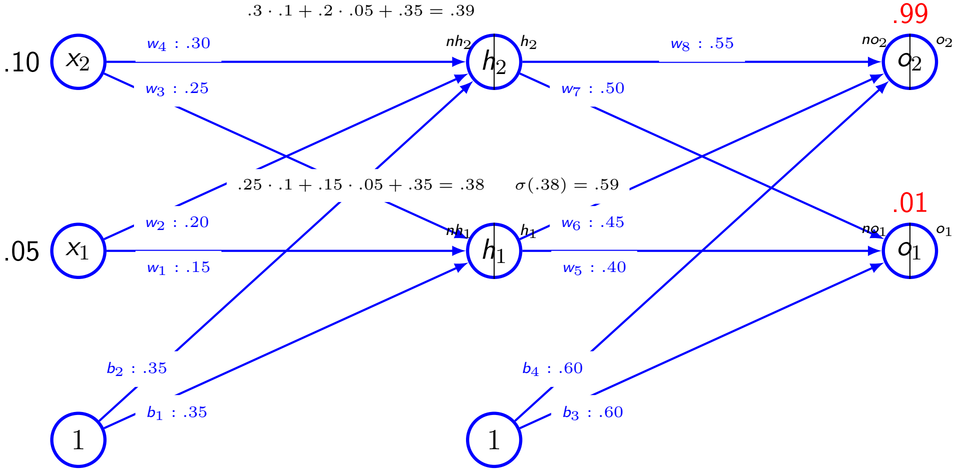
.99
.01
.10
.05
x_2
x_1
1
1
w_4 : .30
w_3 : .25
w_2 : .20
w_1 : .15
b_2 : .35
b_1 : .35
nh_2
h_2
nh_1
h_1
w_8 : .55
w_7 : .50
w_6 : .45
w_5 : .40
b_4 : .60
b_3 : .60
no_2
o_2
no_1
o_1
$.25 \cdot .1 + .15 \cdot .05 + .35 = .38$
$\sigma(0.38)$

Example



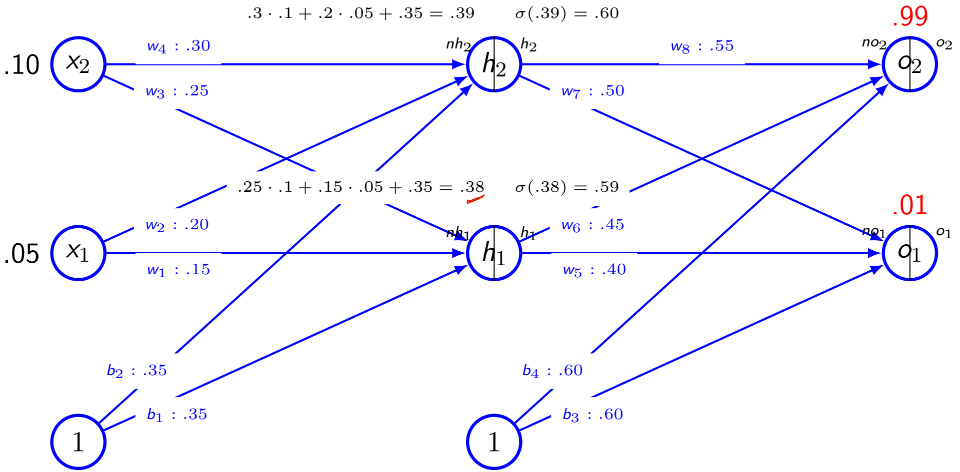
.10
.05
x_2
x_1
1
1
w_4 : .30
w_3 : .25
w_2 : .20
w_1 : .15
b_2 : .35
b_1 : .35
nh_2
h_2
nh_1
h_1
w_8 : .55
w_7 : .50
w_6 : .45
w_5 : .40
b_4 : .60
b_3 : .60
no_2
o_2
no_1
o_1
.99
.01
$.25 \cdot .1 + .15 \cdot .05 + .35 = .38$
$\sigma(.38) = .59$

Example



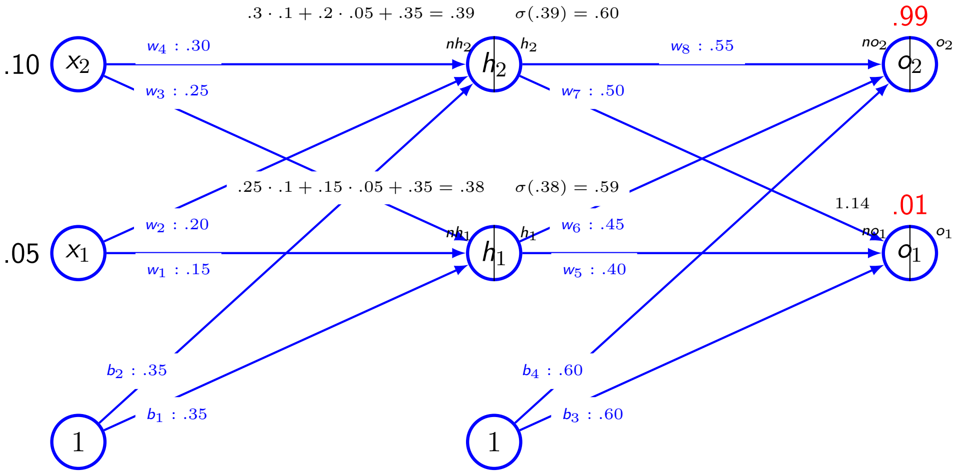
.3 · .1 + .2 · .05 + .35 = .39
.99
.10
x_2
w_4 : .30
nh_2
h_2
h_2
w_8 : .55
no_2
o_2
o_2
w_3 : .25
w_7 : .50
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
.01
.05
x_1
w_2 : .20
nh_1
h_1
h_1
w_6 : .45
no_1
o_1
o_1
w_1 : .15
w_5 : .40
b_2 : .35
b_4 : .60
b_1 : .35
b_3 : .60
1
1

Example

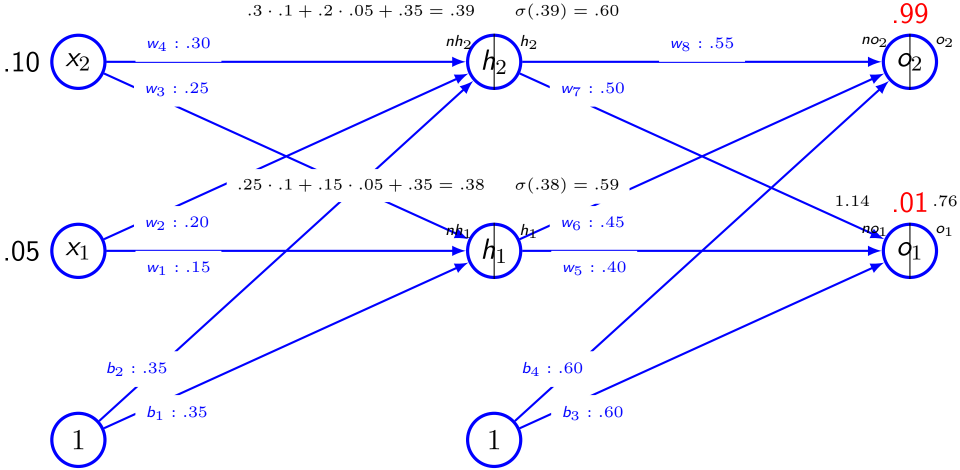


.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
.99
.10
x2
w4 : .30
w3 : .25
nh2
h2
w8 : .55
w7 : .50
no2
o2
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
.01
.05
x1
w2 : .20
w1 : .15
nh1
h1
w6 : .45
w5 : .40
no1
o1
b2 : .35
b1 : .35
b4 : .60
b3 : .60
1
1

Example

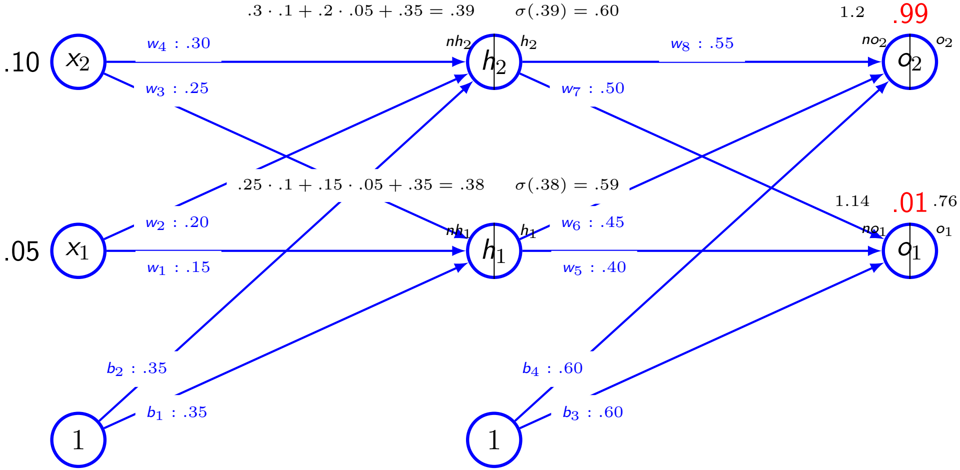

.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
.99
.10
x2
w4 : .30
w3 : .25
nh2
h2
h2
w8 : .55
w7 : .50
no2
o2
o2
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
1.14
.01
.05
x1
w2 : .20
w1 : .15
nh1
h1
h1
w6 : .45
w5 : .40
no1
o1
o1
b2 : .35
b1 : .35
b4 : .60
b3 : .60
1
1

Example



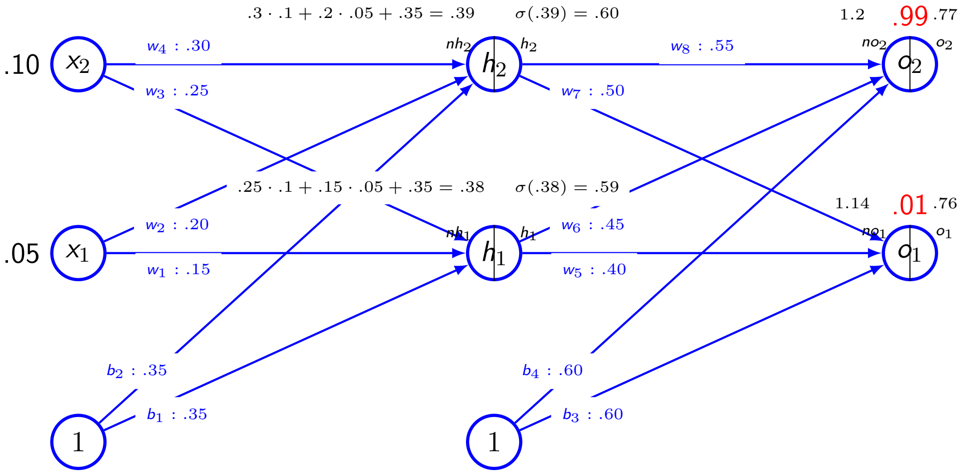
$.3 \cdot .1 + .2 \cdot .05 + .35 = .39$
$\sigma(.39) = .60$
.99
.10
x_2
$w_4 : .30$
nh_2
h_2
$w_8 : .55$
no_2
o_2
$w_3 : .25$
$w_7 : .50$
$.25 \cdot .1 + .15 \cdot .05 + .35 = .38$
$\sigma(.38) = .59$
1.14
.01
.76
$w_2 : .20$
$w_6 : .45$
nh_1
h_1
no_1
o_1
.05
x_1
$w_1 : .15$
$w_5 : .40$
$b_2 : .35$
$b_4 : .60$
$b_1 : .35$
$b_3 : .60$
1
1

Example

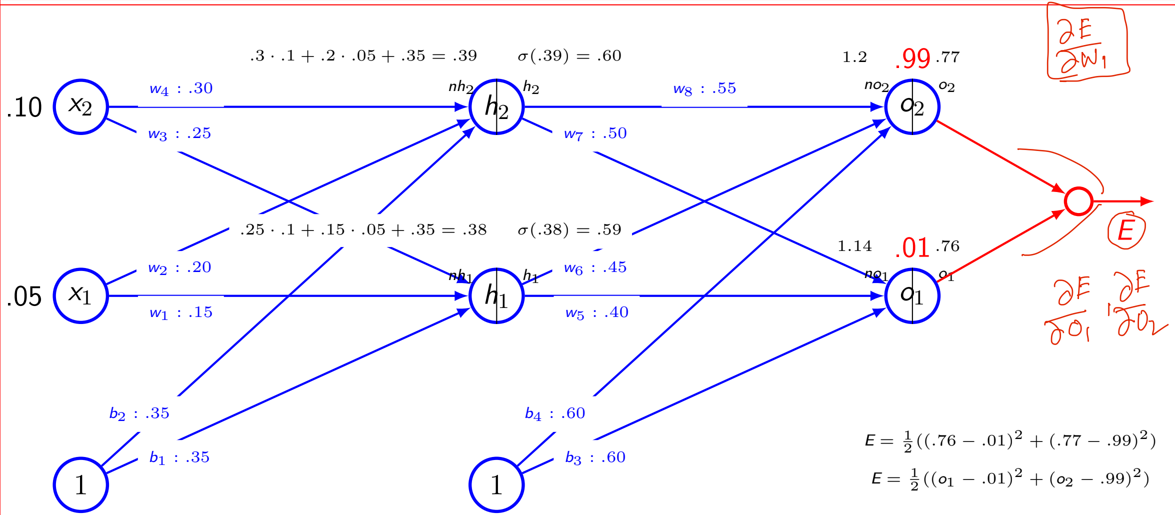


.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
1.2
.99
.10
x2
w4 : .30
w3 : .25
nh2
h2
w8 : .55
w7 : .50
no2
o2
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
1.14
.01
.76
.05
x1
w2 : .20
w1 : .15
nh1
h1
w6 : .45
w5 : .40
no1
o1
b2 : .35
b1 : .35
b4 : .60
b3 : .60
1
1

Example


.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
1.2
.99
.77
.10
x2
w4 : .30
w3 : .25
nh2
h2
h2
w8 : .55
w7 : .50
no2
o2
o2
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
1.14
.01
.76
.05
x1
w2 : .20
w1 : .15
nh1
h1
h1
w6 : .45
w5 : .40
no1
o1
o1
b2 : .35
b1 : .35
1
b4 : .60
b3 : .60
1

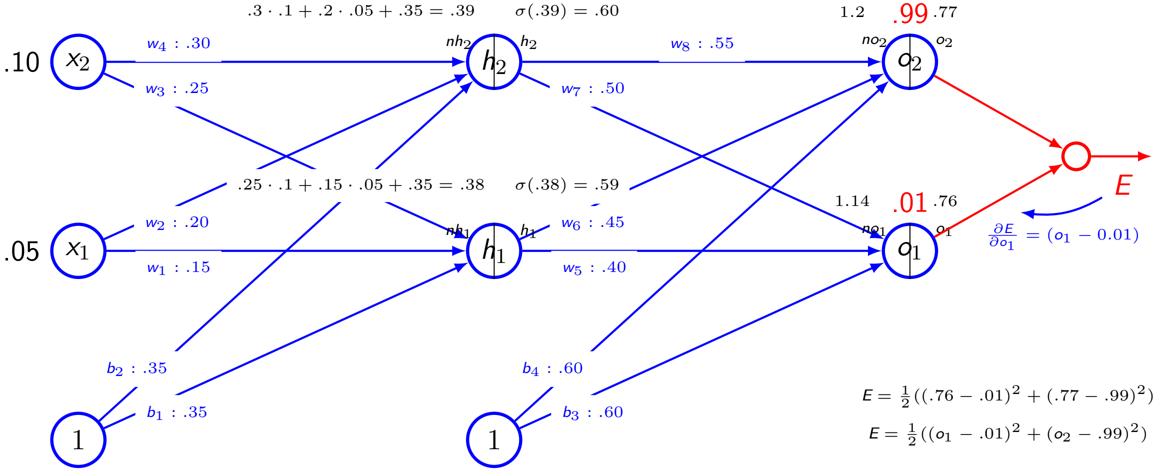
Example



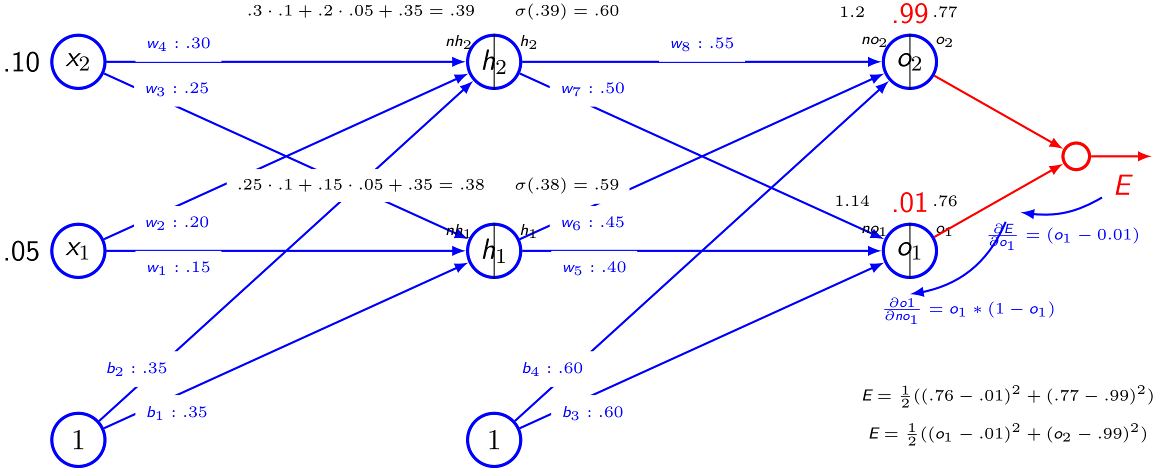
$$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$$

$$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$$

Example


.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
1.2
.99
.77
.10
x_2
w_4 : .30
nh_2
h_2
w_8 : .55
no_2
o_2
w_3 : .25
w_7 : .50
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
E
1.14
.01
.76
$\frac{\partial E}{\partial o_1} = (o_1 - 0.01)$
.05
x_1
w_2 : .20
nh_1
h_1
w_6 : .45
no_1
o_1
w_1 : .15
w_5 : .40
b_2 : .35
b_4 : .60
b_1 : .35
b_3 : .60
1
1
$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$
$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$

Example



$.3 \cdot .1 + .2 \cdot .05 + .35 = .39$
$\sigma(.39) = .60$
1.2
.99
.77
.10
x_2
$w_4 : .30$
$w_3 : .25$
nh_2
h_2
h_2
$w_8 : .55$
$w_7 : .50$
no_2
o_2
o_2
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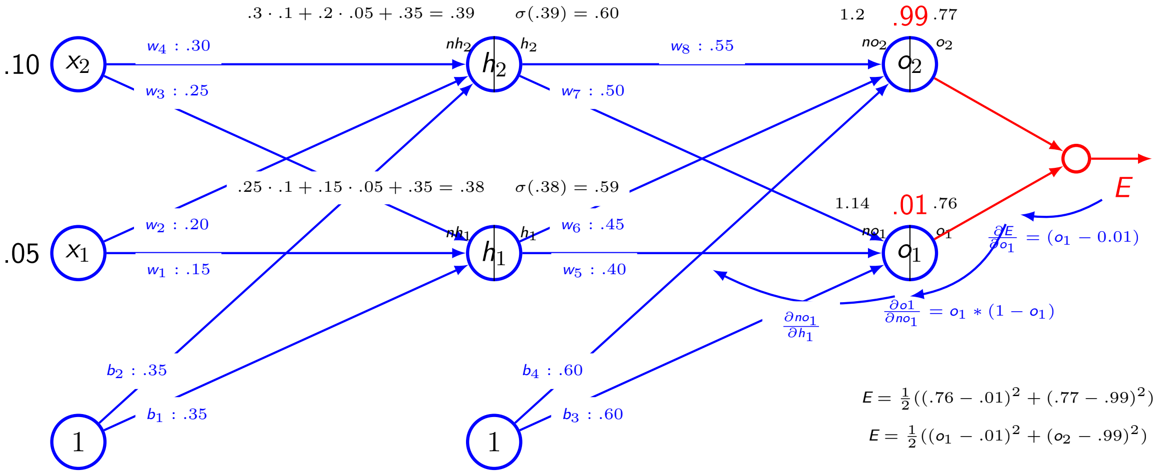
1.14
.01
.76
E
.05
x_1
$w_2 : .20$
$w_1 : .15$
nh_1
h_1
h_1
$w_6 : .45$
$w_5 : .40$
no_1
o_1
o_1
$\frac{\partial E}{\partial o_1} = (o_1 - 0.01)$
$\frac{\partial o1}{\partial no_1} = o_1 * (1 - o_1)$
$b_2 : .35$
$b_1 : .35$
1
$b_4 : .60$
$b_3 : .60$
1
$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$
$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$

Example



.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
1.2
.99
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.10
x2
w4 : .30
nh2
h2
h2
w8 : .55
no2
o2
o2
w3 : .25
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.25 · .1 + .15 · .05 + .35 = .38
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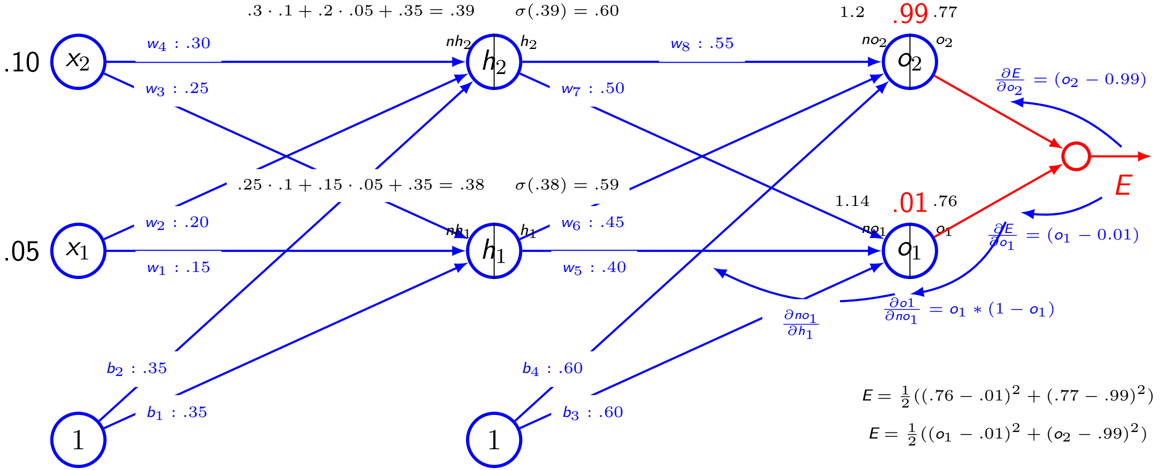
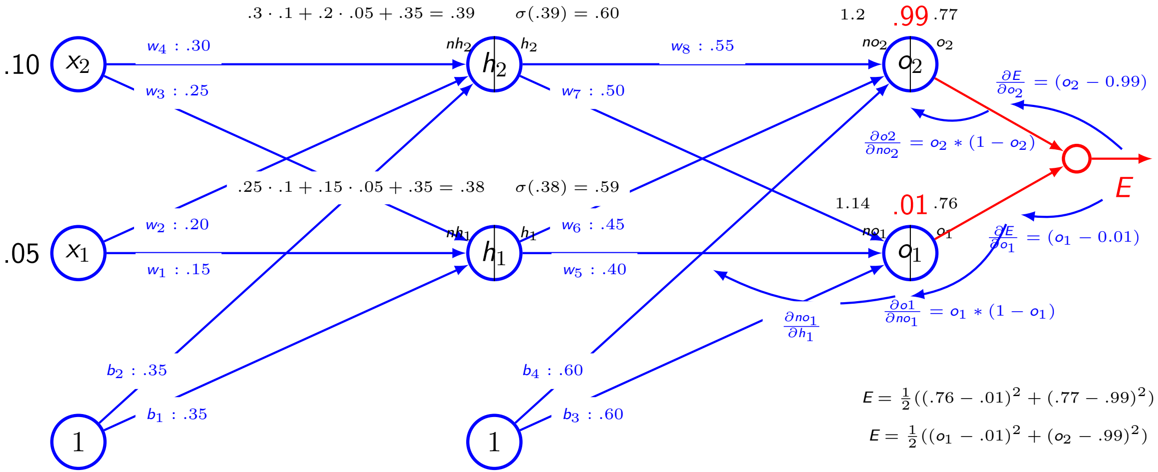
E
1.14
.01
.76
.05
x1
w2 : .20
nh1
h1
h1
w6 : .45
no1
o1
o1
∂E/∂o1 = (o1 − 0.01)
w1 : .15
w5 : .40
∂o1/∂no1 = o1 ∗ (1 − o1)
∂no1/∂h1
b2 : .35
b4 : .60
b1 : .35
b3 : .60
1
1
E = 1/2((.76 − .01)^2 + (.77 − .99)^2)
E = 1/2((o1 − .01)^2 + (o2 − .99)^2)

Example



.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
1.2
.99
.77
.10
x2
w4 : .30
nh2
h2
w8 : .55
no2
o2
∂E/∂o2 = (o2 − 0.99)
w3 : .25
w7 : .50
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
1.14
.01
.76
E
.05
x1
w2 : .20
nh1
h1
w6 : .45
no1
o1
∂E/∂o1 = (o1 − 0.01)
w1 : .15
w5 : .40
∂no1/∂h1
∂o1/∂no1 = o1 ∗ (1 − o1)
b2 : .35
b4 : .60
b1 : .35
b3 : .60
1
1
E = 1/2((.76 − .01)^2 + (.77 − .99)^2)
E = 1/2((o1 − .01)^2 + (o2 − .99)^2)

Example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1.2
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.10
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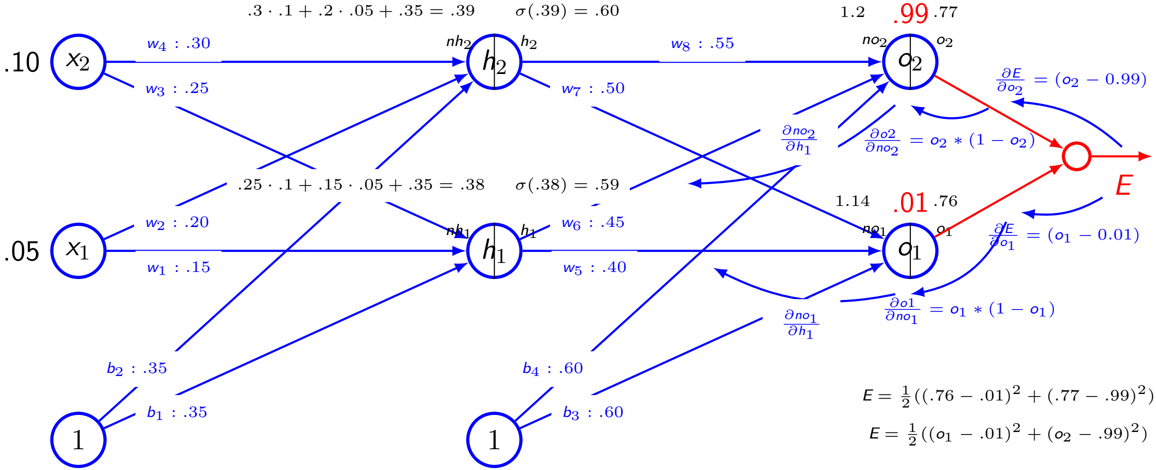
h_2
h_2
w_8 : .55
no_2
o_2
o_2
w_3 : .25
w_7 : .50
$\frac{\partial E}{\partial o_2} = (o_2 - 0.99)$
$\frac{\partial o2}{\partial no_2} = o_2 * (1 - o_2)$
E
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
1.14
.01
.76
w_2 : .20
w_6 : .45
nh_1
h_1
no_1
o_1
$\frac{\partial E}{\partial o_1} = (o_1 - 0.01)$
.05
x_1
h_1
o_1
w_1 : .15
w_5 : .40
$\frac{\partial o1}{\partial no_1} = o_1 * (1 - o_1)$
$\frac{\partial no_1}{\partial h_1}$
b_2 : .35
b_4 : .60
b_1 : .35
b_3 : .60
1
1
$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$
$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$

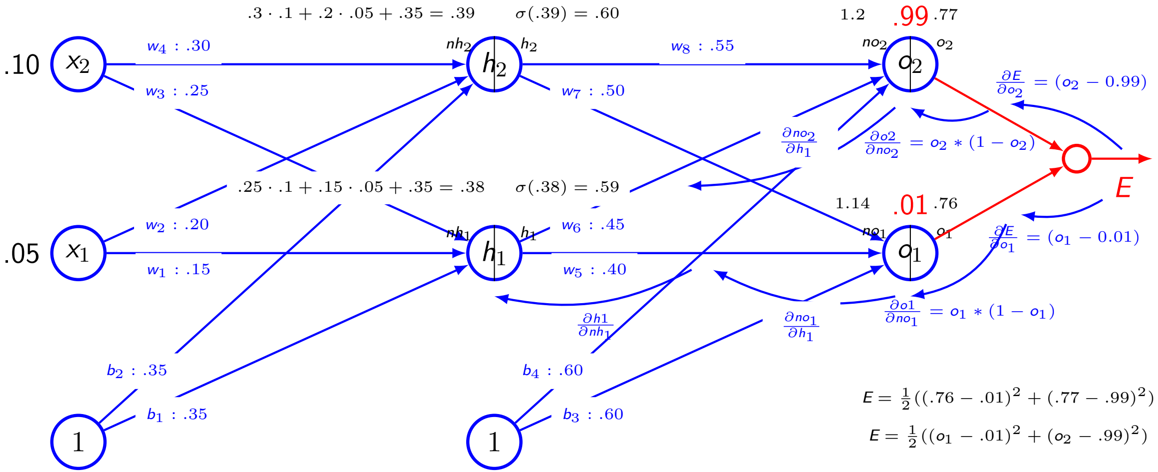
Example



$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$

$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$

Example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.10
x2
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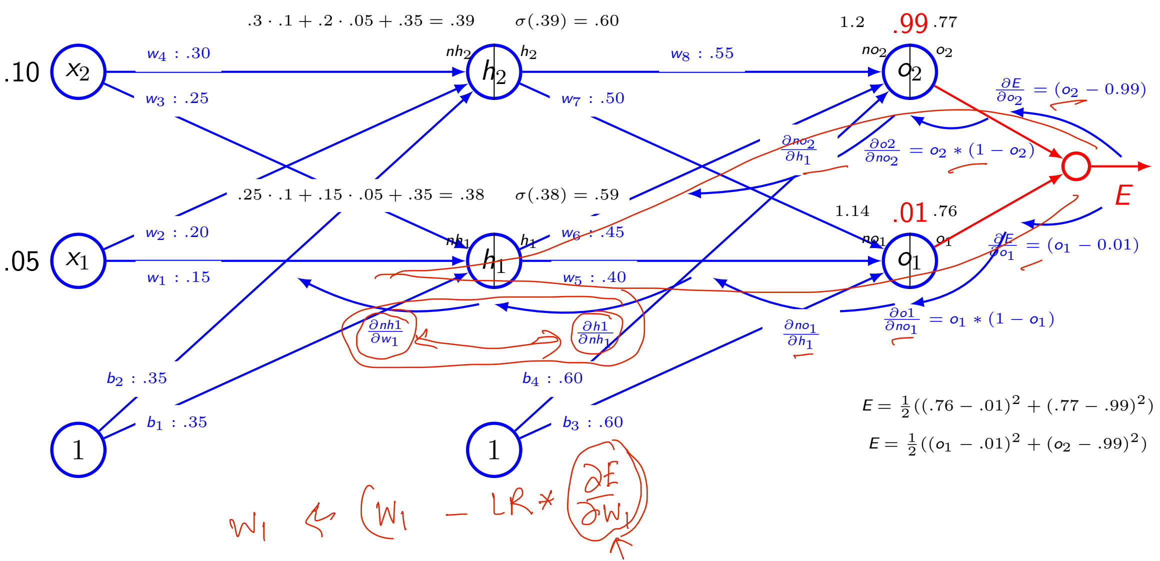
w3 : .25
nh2
h2
h2
w8 : .55
w7 : .50
no2
o2
o2
∂E/∂o2 = (o2 − 0.99)
∂no2/∂h1
∂o2/∂no2 = o2 ∗ (1 − o2)
.25 · .1 + .15 · .05 + .35 = .38
σ(.38) = .59
1.14
.01
.76
E
.05
x1
w2 : .20
w1 : .15
nh1
h1
h1
w6 : .45
w5 : .40
no1
o1
o1
∂E/∂o1 = (o1 − 0.01)
∂h1/∂nh1
∂no1/∂h1
∂o1/∂no1 = o1 ∗ (1 − o1)
b2 : .35
b1 : .35
b4 : .60
b3 : .60
1
1
E = ½((.76 − .01)² + (.77 − .99)²)
E = ½((o1 − .01)² + (o2 − .99)²)

Example



.3 · .1 + .2 · .05 + .35 = .39
σ(.39) = .60
1.2
.99
.77
.10
x_2
w_4 : .30
nh_2
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w_8 : .55
no_2
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w_3 : .25
w_7 : .50
$\frac{\partial E}{\partial o_2} = (o_2 - 0.99)$
$\frac{\partial no_2}{\partial h_1}$
$\frac{\partial o_2}{\partial no_2} = o_2 * (1 - o_2)$
.25 · .1 + .15 · .05 + .35 = .38
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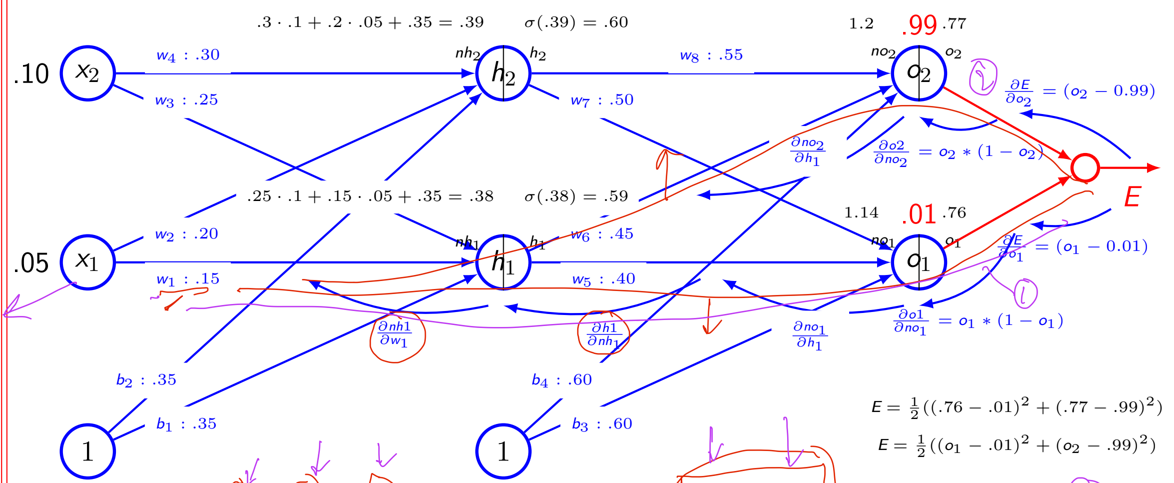
w_1 : .15
w_5 : .40
$\frac{\partial nh1}{\partial w_1}$
$\frac{\partial h1}{\partial nh_1}$
$\frac{\partial no_1}{\partial h_1}$
$\frac{\partial o1}{\partial no_1} = o_1 * (1 - o_1)$
b_2 : .35
b_4 : .60
b_1 : .35
b_3 : .60
1
1
$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$
$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$
$W_1 \leftarrow (W_1 - LR * \frac{\partial E}{\partial W_1})$

Example



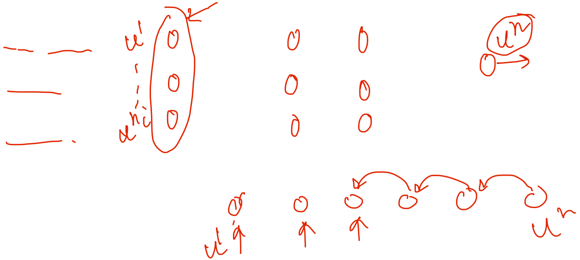
$$E = \frac{1}{2}((.76 - .01)^2 + (.77 - .99)^2)$$

$$E = \frac{1}{2}((o_1 - .01)^2 + (o_2 - .99)^2)$$

$$\frac{\partial E}{\partial w_1} = \left(\frac{\partial E}{\partial o_1} \times \frac{\partial o_1}{\partial no_1} \times \frac{\partial no_1}{\partial h1} + \frac{\partial E}{\partial o_2} \times \frac{\partial o_2}{\partial no_2} \times \frac{\partial no_2}{\partial h1}\right) \times \frac{\partial h1}{\partial nh1} \times \frac{\partial nh1}{\partial w_1}$$

Application of chain rule

- Let us consider $u^{(n)}$ be the loss quantity. Need to find out the gradient for this.
- Let $u^{(1)}$ to $u^{(n_i)}$ are the inputs
- Therefore, we wish to compute $\frac{\partial u^{(n)}}{\partial u^{(i)}}$ where $i = 1, 2, \ldots, n_i$
- Let us assume the nodes are ordered so that we can compute one after another
- Each $u^{(i)}$ is associated with an operation $f^{(i)}$ ie. $u^{(i)} = f(\mathbb{A}^{(i)})$



Algorithm for forward pass

for $i = 1, \ldots, n_i$ **do**

  $u^{(i)} \leftarrow x_i$

end for

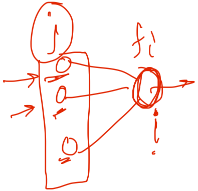
for $i = n_i + 1, \ldots, n$ **do**

  $\mathbb{A}^{(i)} \leftarrow \{u^{(j)} | j \in Pa(u^{(i)})\}$

  $u^{(i)} \leftarrow f^{(i)}(\mathbb{A}^{(i)})$

end for

return $u^{(n)}$



Algorithm for backward pass

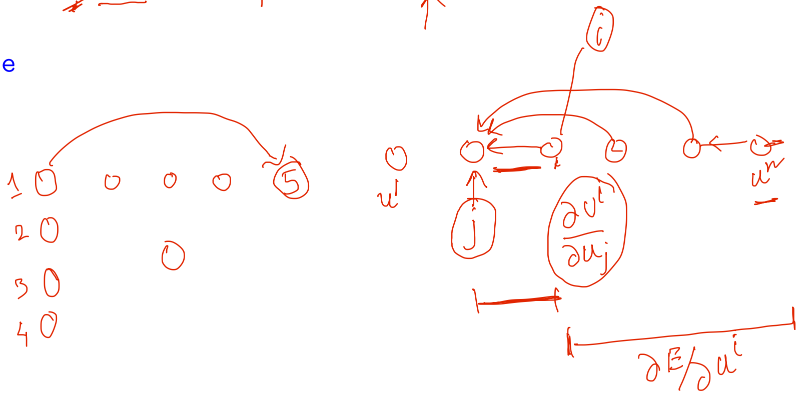
$\texttt{grad\_table}[u^{(n)}] \leftarrow 1$

for $j = n-1$ down to 1 **do**

$$\texttt{grad\_table}[u^{(j)}] \leftarrow \sum_{i:j \in Pa(u^{(i)})} \texttt{grad\_table}[u^{(i)}] \frac{\partial u^{(i)}}{\partial u^{(j)}}$$

end for

return grad_table



Computational graph & subexpression

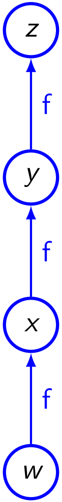
- We have $x = f(w)$, $y = f(x)$, $z = f(y)$

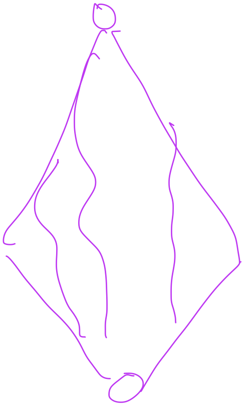
$$\frac{\partial z}{\partial w}$$

$$= \frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial w}$$

$$= f'(y)f'(x)f'(w)$$

$$= f'(f(f(w)))f'(f(w))f'(w)$$



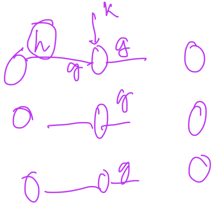


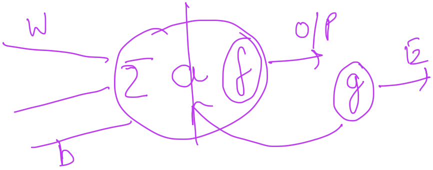
Forward propagation in MLP

- Input
  - $\mathrm{h}^{(0)} = \mathrm{x}$
- Computation for each layer $k = 1, \ldots, l$
  - $\mathrm{a}^{(k)} = \mathrm{b}^{(k)} + \mathrm{W}^{(k)} \mathrm{h}^{(k-1)}$
  - $\mathrm{h}^{(k)} = f(\mathrm{a}^{(k)})$
- Computation of output and loss function
  - $\hat{\mathrm{y}} = \mathrm{h}^{(l)}$
  - $J = L(\hat{\mathrm{y}}, \mathrm{y}) + \lambda \Omega(\theta)$

Backward computation in MLP

- Compute gradient at the output
  - $g \leftarrow \nabla_{\hat{y}} J = \nabla_{\hat{y}} L(\hat{y}, y)$
- Convert the gradient at output layer into gradient of pre-activation
  - $g \leftarrow \nabla_{a^{(k)}} J = g \odot f'(a^{(k)})$
- Compute gradient on weights and biases
  - $\nabla_{b^{(k)}} J = g + \lambda \nabla_{b^{(k)}} \Omega(\theta)$
  - $\nabla_{W^{(k)}} J = g h^{(k-1)T} + \lambda \nabla_{W^{(k)}} \Omega(\theta)$
- Propagate the gradients wrt the next lower level activation
  - $g \leftarrow \nabla_{h^{(k-1)}} J = W^{(k)T} g$

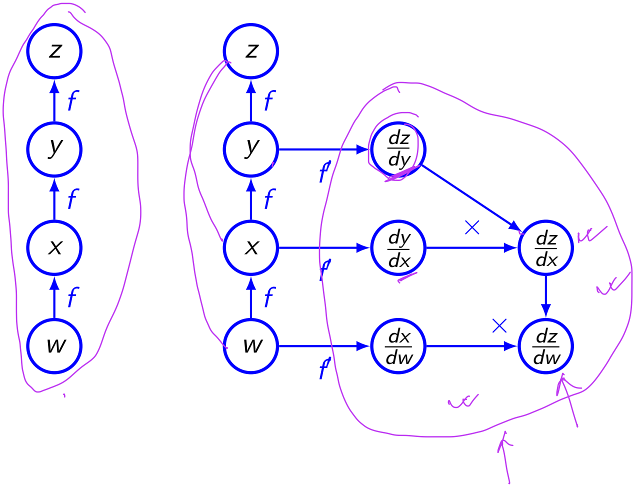




Computation of derivatives

- Takes a computational graph and a set of numerical values for the inputs, then return a set of numerical values
  - Symbol-to-number differentiation
  - Torch, Caffe
- Takes computational graph and add additional nodes to the graph that provide symbolic description of derivative
  - Symbol-to-symbol derivative
  - Theano, TensorFlow

Example



Summary

- Writing gradient for each parameter is difficult
- Recursive application of chain rule along the computational graph help to compute the gradients
- Forward pass - compute the value of the operations and store the necessary information
- Backward pass - uses the loss function, computes the gradient, updates the parameters.