Discrete Mathematics

Planar Graph

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Example

- **Three sworn enemies A, B, C live in houses in the woods. We must cut paths so that each has a path to each of the tree utilities - gas, water, electricity. In order to avoid confrontations, we do not want any of the paths to cross. Can this be done?**

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  - Answer is NO

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- **A plane graph is a particular planar graph embedding of a planar graph**
- **A planar embedding of a graph cuts the plane into pieces.**
- **The faces of a plane graph are the maximal regions of the plane that contain no point used in the embedding**

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- Proof: We use induction on n
  - For $n = 1$: If $e = 0$ then $f = 1$ and the formula holds.
  - Each added loop passes through a face and cuts it into two faces. This augments the edge count and face count each by 1. Thus the formula holds when $n = 1$ for any number of edges.
  - Induction step, $n > 1$: Since G is connected, we can find an edge that is not a loop. When we contract such and edge, we obtain a plane graph G' with n' vertices, e' edges, and f' faces.

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  - So, we have $n' = n - 1$, $e' = e - 1$ and $f' = f$.
  - Hence, $n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2$

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- **Check K_5 and $K_{3,3}$**

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- **In a graph G, subdivision of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex**
- **If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is non-planar.**
- **A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$**

Thank you!