Discrete Mathematics

Trees

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- A spanning tree is a spanning subgraph that is a tree

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- **A graph that is a tree has exactly one spanning tree, the full graph itself.**
- **A spanning subgraph of G need not be connected, and a connected subgraph of G need not be a spanning subgraph.**

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  - Thus G' is a tree with $n-1$ vertices.

Tree properties-II

- **For an n-vertex graph G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vertices)**
  - **G is connected and has no cycles**
  - **G is connected and has $n-1$ edges**
  - **G has $n-1$ edges and no cycles**
  - **For $u, v \in V(G)$, G has exactly one u, v-path**

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- Adding one edge to a tree forms exactly one cycle
- Every connected graph contains a spanning tree

Tree properties-IV

- **If T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.**

Distance

- If G has a u, v-path, then the distance from u to v, written $d(u, v)$ is the least length of a u, v-path.
  - If G has no such path, then $d(u, v) = \infty$.
- The diameter is $\max_{u,v \in V(G)} d(u, v)$

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- **Prufer code:**
  - **Input: A tree T with vertex set $[n]$**
  - **Output: $f(T) = (a_1, a_2, \ldots, a_{n-2})$**
  - **Iteration: At the ith step, delete the least remaining leaf, and let a_i be the neighbor of this leaf. Note a_i**

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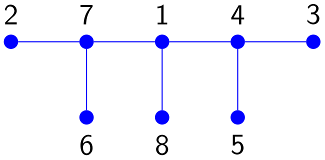

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  - $\dfrac{(n-2)!}{\prod (d_i - 1)!}$

Thank you!