Discrete Mathematics

Graphs-I

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- **Example**
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  - **How can you prepare semester examination schedule for IIT Patna?**
  - **JEE seat allocation!!**

Graphs

- A graph $G = (V, E)$ with m vertices and n edges consists of
  - A vertex set $V(G) = \{v_1, v_2, \ldots, v_m\}$
  - An edge set $E(G) = \{e_1, e_2, \ldots, e_n\}$ where $e_i = (v_k, v_{k'})$
    - Here, v_k and $v_{k'}$ are the two end points of the edge
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  - No multiple edges
    - Multiple edges are the edges with same pair of end points.

Definition-I

- **Complement graph**
  - **The complement graph $\bar{G}$ of a simple graph G is simple graph with vertex set $V(G)$ and $(u, v) \in E(\bar{G})$ if and only if $(u, v) \notin E(G)$, where $u, v \in V(G)$**

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- **Clique**
  - **A clique in a graph is a set of pairwise adjacent vertices**
- **Independent set**
  - **An independent set in a graph is a set of pairwise non-adjacent vertices**

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- **Chromatic number**
  - **The chromatic number of a graph G, denoted as $\chi(G)$ is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors**
  - **A graph is k-partite if $V(G)$ can be expressed as the union of k independent sets.**

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- **Cycle**
  - **A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle**

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- **Subgraph**
  - **A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. This is denoted as $H \subseteq G$ and we say 'G contains H'.**

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- **Connected graph**
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- **Isomorphism**
  - **An isomorphism from a simple graph G to a simple graph H is a bijection $f: V(G) \rightarrow V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$. We say G is isomorphic to H and denoted as $G \cong H$, if there is an isomorphism from G to H**

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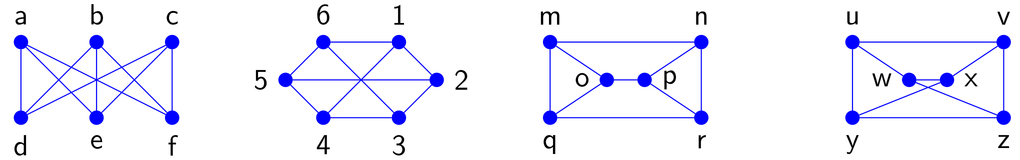
  - **Isomorphic relation is an equivalence relation on the set of simple graph**
    - **It satisfies reflexive, symmetric, and transitive properties.**

Example-I

- **Which of the following graphs are isomorphic**



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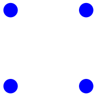

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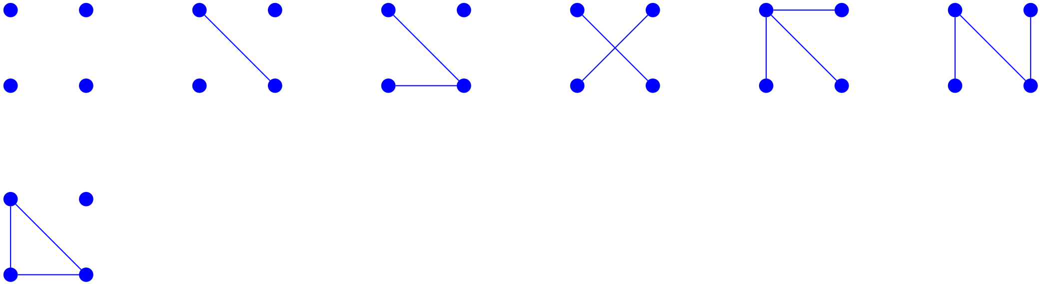

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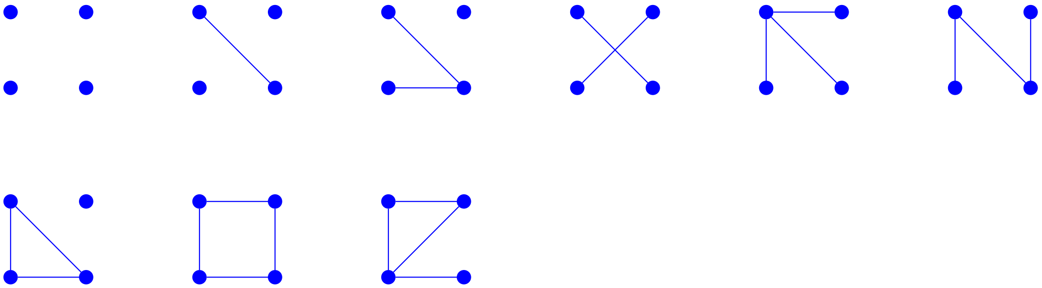

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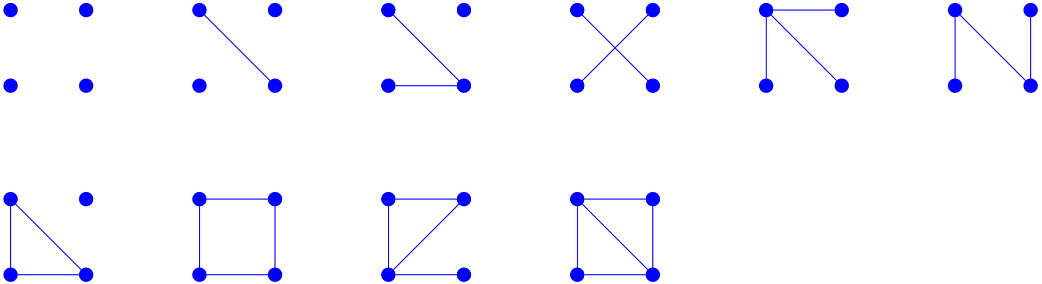

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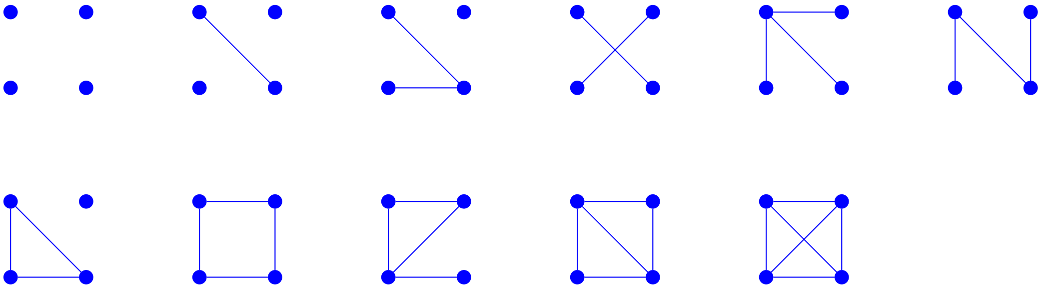

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  - $K_{n,m}$ **- complete bipartite graph with with sets having** n **and** m **vertices**

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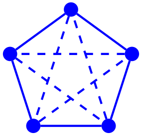
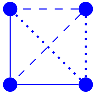
- **Decomposition**
  - **A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list**





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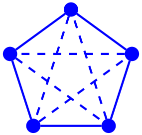
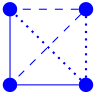
- A graph is self-complementary if it is isomorphic to its complement





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- Let G be the path with vertex set $\{1, 2, 3, 4\}$ and edge set $\{12, 23, 34\}$. How many automorphism are there for this graph?
- Count the number of automorphism for the graph $K_{r,s}$.

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  - By induction hypothesis, W' contains a u, v-path P and this path is contained in W

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- **Component**
  - The components of a graph G are its maximal connected subgraphs.
  - A component is trivial if it has no edges, otherwise it is nontrivial.
  - An isolated vertex is a vertex of degree 0

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  - **Adding an edge decreases the number of components by 0 or 1.**
  - **Adding k edges can reduce the number of components by maximum of k. Hence the number of components is at least $n - k$**

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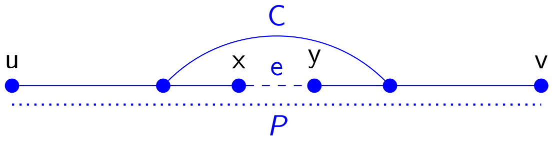

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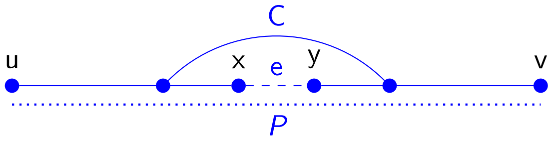
  - If P does not contain e, then P exists in $H - e$



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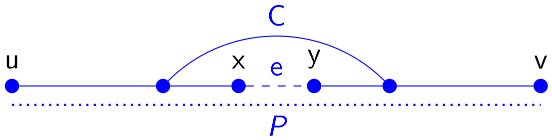
  - **If P contains e, suppose by symmetry that x is between u and y on P. Since $H - e$ contains a u, x-path along P, an x, y-path along C, and a y, v-path along P, the transitivity of connection relation implies that $H - e$ has a u, v-path**



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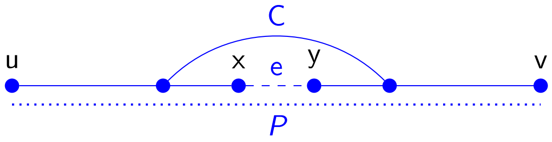
  - If P contains e, suppose by symmetry that x is between u and y on P. Since $H - e$ contains a u, x-path along P, an x, y-path along C, and a y, v-path along P, the transitivity of connection relation implies that $H - e$ has a u, v-path
  - Hence, $H - e$ is connected



Example-VI

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  - **By induction hypothesis it contains an odd cycle.**

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- **Prove: A graph is bipartite if and only if it has no odd cycle**

Thank you!