Discrete Mathematics

Counting

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Combinatorics

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- Counting the number of arrangements with particular property is one of the primary concerns here
- Analyzing complexity of algorithms, discrete probabilities, etc. require knowledge of number of arrangements

Product rule

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  - How many functions are there from a set with m elements to a set with n elements?

Sum rule

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  - The alphabet of a given language consists of three symbols namely A, B, and C. A word in the given language can contain at most 4 letters. How many words are there in the given language?
  - How many ways are there to put one white king and one black king on a chess-board such that they do not attack each other?

Principle of inclusion-exclusion

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- **Example:**
  - **How many bit strings of length eight either start with a 1 bit or end with the two bits 00?**
  - **There are six letters in a given language. A word is a sequence of six letters, some pair of which are the same. How many words are there in the given language?**

Example: Euler's ϕ function

- For $n \in \mathrm{Z}^+$, $n > 2$, let $\phi(n)$ be the number of positive integers m, where $1 \leq m < n$ and $gcd(m, n) = 1$, that is, m, n are relatively prime. Find $\phi(n)$

Example

- **How many ways are there to put eight rooks on a chessboard so that they do not attack each other?**

Example

- There are N boys and N girls in a dance class. How many ways are there to arrange them in pairs for a dance?

The Pigeonhole Principle

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- **Example:**
  - **In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.**
  - **Show that for every integer n there is a multiple of n that has only 0s and 1s in its decimal expansion.**

Exercise

- **During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.**

Exercise

- **Show that an equilateral triangle cannot be covered completely by two smaller equilateral triangles.**

Exercise

- **Prove that there exists a power of three which ends with the digits 001 in decimal notation**

Permutation

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- **A permutation of a set of distinct objects is an ordered arrangement of the objects.**
- **An ordered permutation of r elements of a set is called a r-permutation.**

- **It is denoted as $P(n, r)$ or ${}^nP_r$**
- $P(n, r) = n \cdot (n-1) \cdots (n-r+1) = \frac{n!}{(n-r)!}$

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- **Also,** $\binom{n}{r} = \binom{n}{n-r}$
- **Other representations:** $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} = \frac{n}{r}\binom{n-1}{r-1}$

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- The number of r-permutations of a set of n objects with repetition allowed is n^r

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- **Example:**
  - **Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?**

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- **Distinguishable objects and indistinguishable boxes**
  - **Let $S(n, j)$ be the number of ways to allocate n distinguishable objects in j indistinguishable boxes such that no box is empty**
$$S(n, j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j - i)^n$$
    - **Stirling number of the second type**

Exercise

- **A group of n fans of the winning IPL team throw their hats high into the air. The hats come back randomly, one hat to each of the n fans. How many ways $h(n, k)$ are there for exactly k fans to their own hats back?**

Exercise

- $2n$ **players are participating in a chess tournament. Find the number P_n of pairings for the first round.**

Exercise

- **Find a closed formula for** $S_n = \sum_{k=1}^{n} \binom{n}{k} k^2$

Exercise

- **Consider all $2^n - 1$ non-empty subsets of the set $\{1, 2, \ldots, n\}$. For every such subset, we find the product of the reciprocals of each of its elements. Find the sum of all these products.**

Exercise

- **Find the number of integers from 0 through 999999 that have no two equal neighboring digits in their decimal representation.**

Exercise

- **A rook stands on the leftmost box of a 1 x 30 strip of squares and can shift any number of boxes to the right in one move.**

  i) How many ways are there for the rook to reach the rightmost box?

  ii) How many ways are there to reach the rightmost box in exactly 7 moves?

Exercise

- **Within a table of m rows and n columns a box is marked at the intersection of the pth row and the qth column. How many of the rectangles formed by the boxes of the table contain the marked box?**

Thank you!