Discrete Mathematics

Relations

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- Functions vs Relations
- How many relations are there on a set with n elements?

Properties: Reflexive

- A relation R on a set A is **reflexive** if $(a, a) \in R$ for every element $a \in A$
- Which of the following are reflexive? (set of integers)
  - $R_1 = \{(a, b) | a \leq b\}$
  - $R_2 = \{(a, b) | a > b\}$
  - $R_3 = \{(a, b) | a = b$ or $a = -b\}$
  - $R_4 = \{(a, b) | a = b\}$
  - $R_5 = \{(a, b) | a = b + 1\}$
  - $R_6 = \{(a, b) | a + b \leq 3\}$

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  - **How to express symmetric/antisymmetric conditions using quantifiers?**

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  - **How to express transitive condition using quantifiers?**

Problem

- **How many reflexive relations are there on a set with n elements?**

Composition of relations

- Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A, c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $S \circ R$.
- Let R be a relation on the set A. The powers R^n are defined as $R^1 = R$ and $R^{n+1} = R^n \circ R$
  - Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$. Find R^2, R^3, R^4

Representation

- **Using matrcies**

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- Using matrcies
- Using directed graph

Closures

- If R is a relation on a set A, then the closure of R with respect to P, if it exists, is the relation S on A with property P that contains R and is a subset of every subset of $A \times A$ containing R with property P.

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  - What will be symmetric closure

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- Equivalence class
- Partition

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  - Is inclusion relation $\subseteq$ a partial ordering on the power set of a set S?

Total orderings

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- $(S, \preccurlyeq)$ is a well-ordered set if it is a poset such that $\preccurlyeq$ is a total ordering and every nonempty subset of S has a least element.

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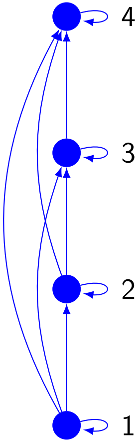

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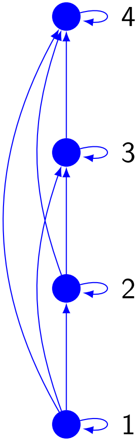

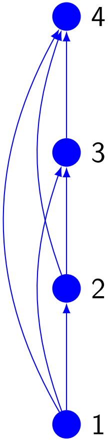

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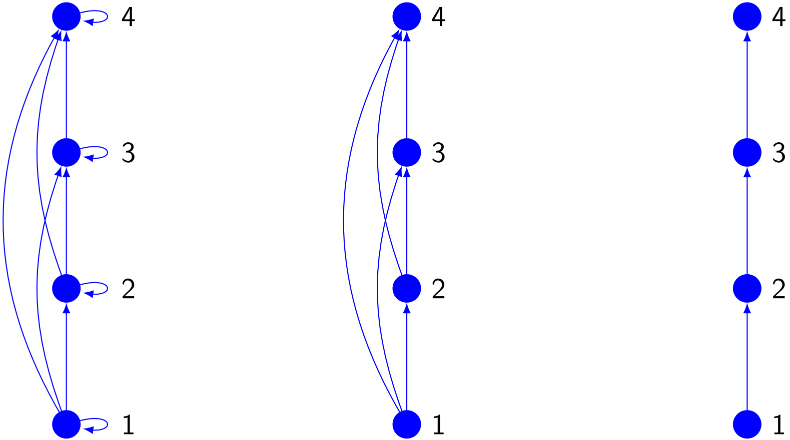

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- Similarly least element
- The element x is called the least upper bound of the subset A if x is an upper bound that is less than every other upper bound of A
- The element y is called the greatest lower bound of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A

Lattices

- A partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound is called a lattice.
  - Is the poset $(\mathbb{Z}^+, |)$ a lattice?
  - Is the poset $(P(S), \subseteq)$ a lattice?

Thank you!