Discrete Mathematics

Propositional Logic: Deduction

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Deduction in Propositional Logic

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- **Conjunction:** a, b **:- therefore** $(a \wedge b)$
- **Addition:** a **:- therefore** $(a \vee b)$
- **Natural deduction is Sound and Complete**

Problem: Glasses

- You are about to leave for college in the morning and discover that you don't have your glasses. You know the following statements are true:
  - If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.
  - If my glasses are on the kitchen table, then I saw them at breakfast.
  - I did not see my glasses at breakfast.
  - I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
  - If I was reading the newspaper in the living room then my glasses are on the coffee table.
- Where are the glasses?

Puzzle-1

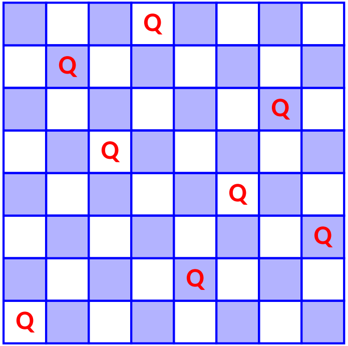
- **As a reward for saving his daughter from pirates, the King has given you the opportunity to win a treasure hidden inside one of three trunks. The two trunks that do not hold the treasure are empty. To win, you must select the correct trunk. Trunks 1 and 2 are each inscribed with the message "This trunk is empty," and Trunk 3 is inscribed with the message "The treasure is in Trunk 2." The Queen, who never lies, tells you that only one of these inscriptions is true, while the other two are wrong. Which trunk should you select to win?**

Puzzle-2

- There is an island that has two kinds of inhabitants, knights, who always tell the truth, and their opposites, knaves, who always lie. You encounter two people A and B. What are A and B if A says "B is a knight" and B says "The two of us are opposite types"?

8-Queens

- Need to place 8 queens such that no two queens attack each other



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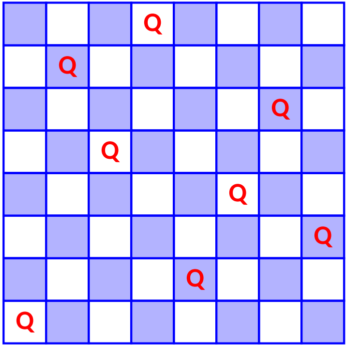

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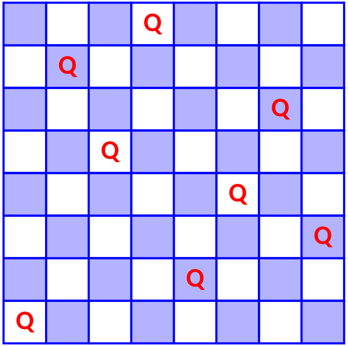

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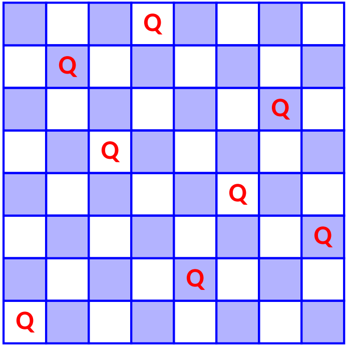

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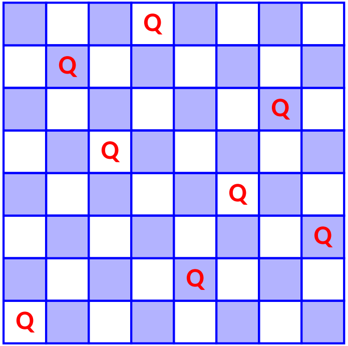

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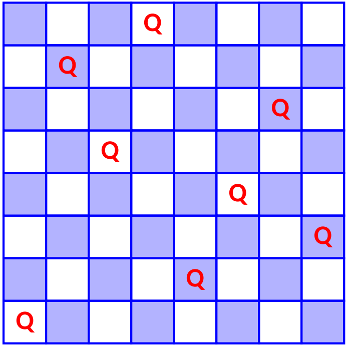

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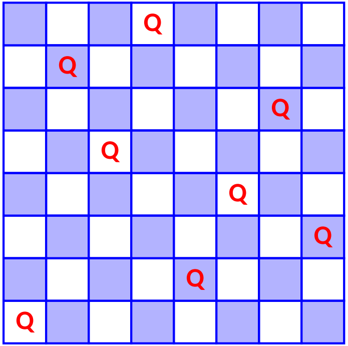

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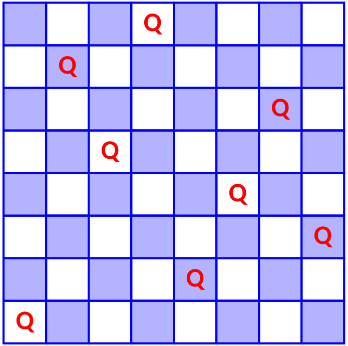
  - $F = F_1 \wedge F_2 \wedge F_3 \wedge F_4 \wedge F_5$



SAT problems

- **Propositions** - $\mathcal{P} = \{a, b, c, \ldots\}$
- **Literals** - $\{a, \neg a, b, \neg b, \ldots\}$
- **Clause** - $C_1 = (a \vee b \vee \neg c)$, $C_2 = (\neg a \vee b \vee \neg d), \ldots$
  - **Clause is disjunction of literals**
- **Formula** - $\mathcal{F} = C_1 \wedge C_2 \wedge \ldots$
  - **Conjunctive normal form (CNF)**
- **Goal is to find an assignment (interpretation) to the propositions such that $\mathcal{F}$ is *true***
  - $\mathcal{F}$ **is satisfiable if there exists at least one valid interpretation**
  - $\mathcal{F}$ **is unsatisfiable if there exists none**

SAT tools

- Very good open-source SAT solvers are available
  - MiniSAT
  - zChaff
  - CaDiCaL
  - Glucose
  - Lingeling
  - PicoSAT
  - Cryptominisat
  - Rsat
  - Riss
  - many others
- `http://www.satcompetition.org/`

Input format - DIMACS

- **There is standard format to specify clauses and its literals**
- **To specify comments**

```
c This line is comment
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- To specify CNF

```
c list_of_literals 0
1 -2 3 0
2 4 0
-3 0
-1 2 3 -4 0
```

Output format

- Outputs from a SAT solver are - SATISFIABLE / UNSATISFIABLE, an assignment of Boolean variables
- Typically it will be as follows

```
SAT
-1 2 -3 4 0
```

  - The last line needs to be interpreted as follows: $\neg a \wedge b \wedge \neg c \wedge d$
  - There may be additional messages to provide information on resource usage, statistics, etc.

SAT modeling: Propositional logic - 1

- If Rajat is the Director then Rajat is well known. Rajat is the Director. So, Rajat is well known.
- Propositions: a : Rajat is the Director, b : Rajat is well known.
- Formula $(\mathcal{F})$: $a \implies b$, a
- Goal $(\mathcal{G})$: b. That is $\mathcal{M} = (\mathcal{F} \implies \mathcal{G}) \equiv ((a \implies b) \wedge a) \implies b$
  - If $\mathcal{M}$ is tautology then $\mathcal{F} \wedge \bar{\mathcal{G}}$ will be false ie. $\mathcal{F} \wedge \mathcal{G} = \emptyset$
  - If $\mathcal{M}$ is satisfiable then so is $\mathcal{F} \wedge \bar{\mathcal{G}}$

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-1 2 0
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```
p cnf 2 3
-1 2 0
1 0
-2 0
```

```
UNSATISFIABLE
```

SAT modeling: Propositional logic - 2

- **If Rajat is the Director then Rajat is well known. Rajat is not the Director. So, Rajat is not well known.**
- **Propositions:** a **: Rajat is the Director,** b **: Rajat is well known.**
- **Formula** $(\mathcal{F})$**:** $a \implies b, \neg a$
- **Goal** $(\mathcal{G})$**:** $\neg b$

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- SAT modeling

```
p cnf 2 3
-1 2 0
-1 0
2 0
```

```
SATISFIABLE
```

Insufficiency of Propositional Logic

- **Wherever Mary goes, so does the lamb. Mary goes to school. So the lamb goes to school.**

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- **No contractors are dependable. Some engineers are contractors. Therefore some engineers are not dependable.**

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- Every passenger is either in first class or second class. Each passenger is in second class if and only if he or she is not wealthy. Some passengers are wealthy. Not all passengers are wealthy. Therefore some passengers are in second class.

Thank you!