

Embedded Systems



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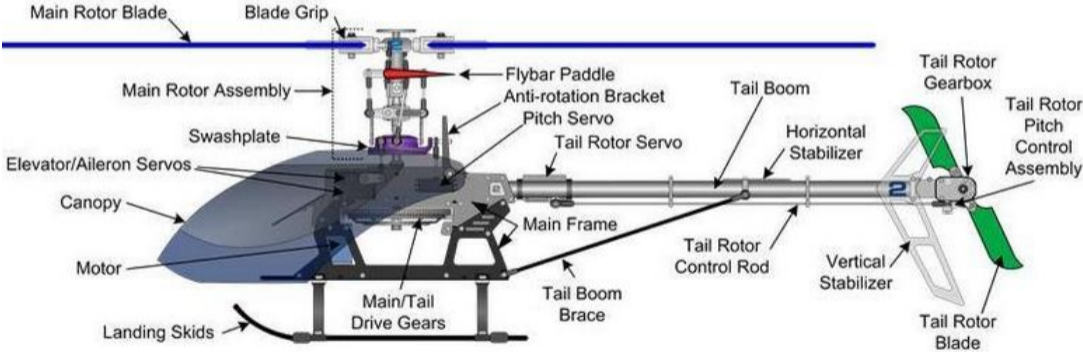
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Modeling: Continuous Systems

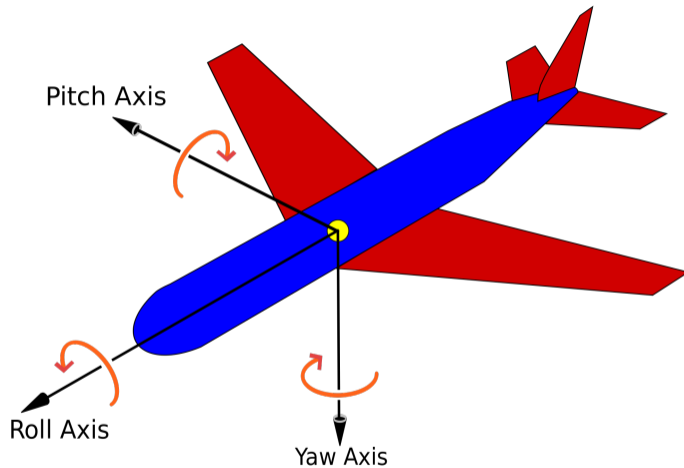
System modeling

- Mimic the real world behavior of the system
- There exist a large variety of systems
 - Mechanical, electrical, chemical, biological, etc.
- Behavior of most of the system can be described using differential equations
- Continuous dynamics
 - Modal models
 - Used for modeling discrete systems
 - For each mode, we have continuous dynamics
- Ordinary differential equation will be used to describe the system
 - Properties like **linearity**, **time invariance**, **stability**, etc. will be considered

Helicopter



Helicopter



Newtonian mechanics

- Motion of object can be represented with six degrees of freedom

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 - Angular motion θ_x (roll), θ_y (yaw), θ_z (pitch)

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- Change in position or orientation can be determined by Newton's 2nd law

$$\mathbf{F}(t) = M\ddot{\mathbf{x}}(t)$$

- \mathbf{F} - force, M - mass and $\ddot{\mathbf{x}}$ - second derivative ie. acceleration

Newtonian mechanics (contd.)

- Solving the equation we get $t > 0$, $\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}(0) + \int_0^t \ddot{\mathbf{x}}(\tau) d\tau$

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- Rotational version of force is torque $\mathbf{T}(t) = \frac{d}{dt} \left(\mathbf{I}(t)\dot{\boldsymbol{\theta}}(t) \right)$
 - I - Moment of inertia (depends on the geometry and orientation)

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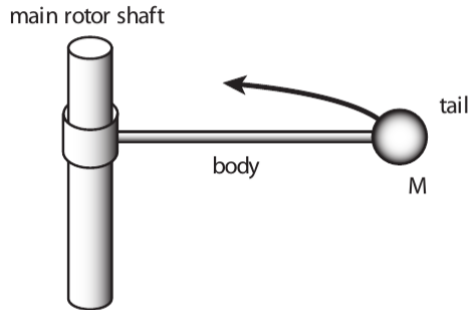
$$\theta(t) = \theta(0) + t\dot{\theta}(0) + \frac{1}{I} \int_0^t \int_0^\tau \mathbf{T}(\alpha) d\alpha d\tau$$

Helicopter model

- Helicopter has two rotors
 - Main rotor to lift
 - Tail rotor to counter balance spin
- Hence, we have

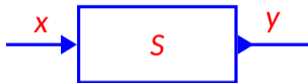
$$\ddot{\theta}_y(t) = T_y(t)/I_{yy} \Rightarrow$$

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau$$



Actor model

- Physical system can be described by input (force, torque) and output (position, orientation, velocity, rotation, etc.)



- Usually X is time (domain) and Y value of particular signal (codomain)
 - $S: X \rightarrow Y, x, y \in \mathbb{R}$

Actor model

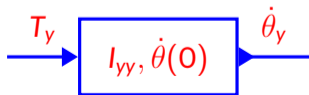
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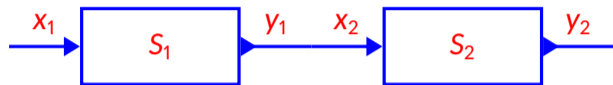
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- Example



Actor model (contd.)

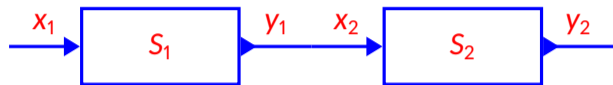
- Actor models are composable



$$\forall t \in \mathbb{R}, \quad y_1(t) = x_2(t)$$

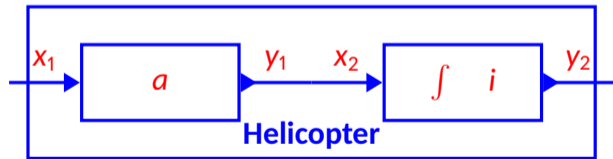
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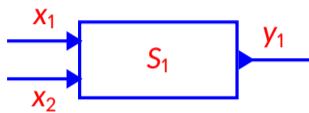
- Example



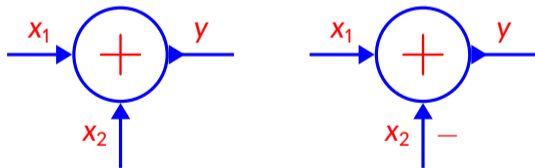
- We have $\forall t \in \mathbb{R} \quad y_2(t) = i + \int_0^t x_2(\tau) d\tau$ where $a = 1/I_{yy}$, $i = \dot{\theta}_y(0)$, $x_1 = T_y$ and $y_2 = \dot{\theta}_y$

Actor model (contd.)

- Actor can have multiple inputs



- Another useful building block is signal adder



- $y(t) = x_1(t) + x_2(t), y(t) = x_1(t) - x_2(t)$

Properties of systems

- Causal system
- Memoryless systems
- Linear and time invariant
- Stability
- Feedback control

Causal systems

- Output depends only on current and past inputs
- Consider a continuous time signal x
- Let $x|_{t \leq \tau}$ represent restriction in time defined only for $t \leq \tau$
- Consider a continuous time system $S : X \rightarrow Y$, the system is causal if for all $x_1, x_2 \in X$ and $\tau \in R$, $x_1|_{t \leq \tau} = x_2|_{t \leq \tau} \Rightarrow S(x_1)|_{t \leq \tau} = S(x_2)|_{t \leq \tau}$
- Strictly causal $\forall \tau \in R$, $x_1|_{t < \tau} = x_2|_{t < \tau} \Rightarrow S(x_1)|_{t \leq \tau} = S(x_2)|_{t \leq \tau}$

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- Example
 - Integrator is strictly causal
 - Adder is not strictly causal but causal
- Strictly causal actors are good for continuous feedback system

Memoryless systems

- A system has memory if the output depends not only on the current inputs but also on the past inputs
- Formally, $S : X \rightarrow Y$ the system is memoryless if there exist a function $f : X \rightarrow Y$ such that for all $x \in X$, $(S(x))(t) = f(x(t))$ for all $t \in R$

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- Example
 - Integrator is not memoryless
 - Adder is memoryless

Linear and time invariant (LTI)

- A systems $S : X \rightarrow Y$ where X and Y are sets of signals is linear if it satisfies the superposition property

$$\forall x_1, x_2 \in X \text{ and } \forall a, b \in R \quad S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$$

- Time invariance means that whether we apply an input to the system now or T seconds from now, the output will be identical except for a time delay of T seconds.
 - Let D_T be the delay operator such that $(D_T(x))(t) = x(t - \tau)$
 - For time invariance, $S(D_T(x)) = D_T(S(x))$

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 - For time invariance, $S(D_T(x)) = D_T(S(x))$

- $\dot{\theta}_y(t) = \frac{1}{I_{yy}} \int_{-\infty}^t T_y(\tau) d\tau$ - LTI

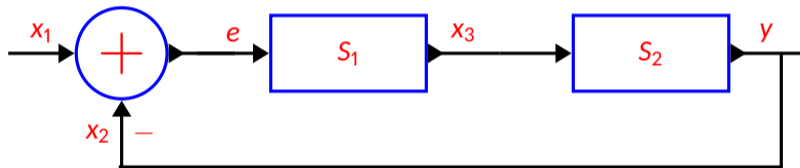
- Many systems are approximated to LTI

Stability

- A system is bounded input bounded output stable if the output signal is bounded for all inputs signals that are bounded
- Helicopter is unstable

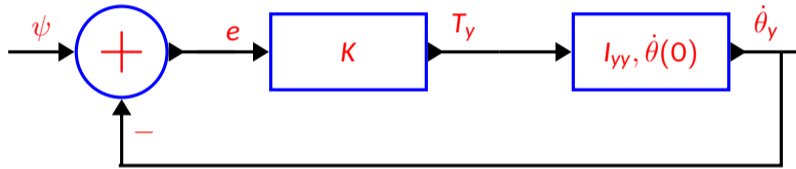
Feedback systems

- A system with feedback has directed cycle where an output from an actor is fed back to affect an input of the same actor



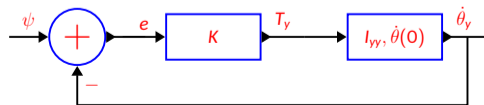
Example: No rotation

- Want to have 0 angular velocity



Example: No rotation (contd.)

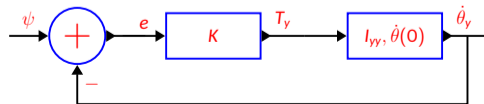
- Our equation remains the same, only input has changed.



Example: No rotation (contd.)

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- $$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau)) d\tau$$

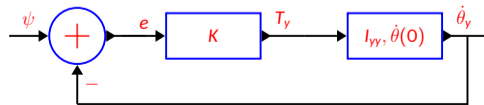


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- We have, $e(t) = \psi(t) - \dot{\theta}_y(t)$, $T_y(t) = Ke(t)$



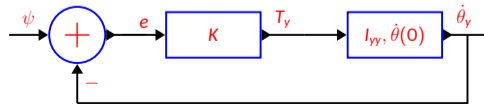
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- Reorganizing we get,
$$\dot{\theta}_y(t) = \dot{\theta}_y(0) - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(t) d\tau$$



Example: No rotation (contd.)

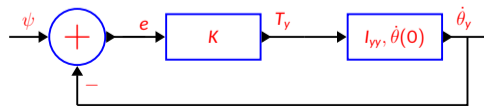
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- Reorganizing we get, $\dot{\theta}_y(t) = \dot{\theta}_y(0) - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau$

- We know, $\int_0^t ae^{a\tau} d\tau = e^{at}u(t) - 1$



Example: No rotation (contd.)

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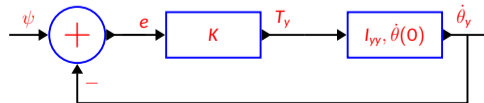
- $\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t (\psi(\tau) - \dot{\theta}_y(\tau)) d\tau$

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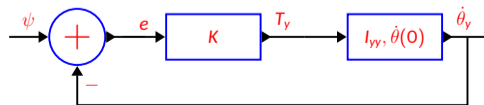
- We know, $\int_0^t ae^{a\tau} d\tau = e^{at}u(t) - 1$

- Therefore we have, $\dot{\theta}_y(t) = \dot{\theta}_y(0)e^{-Kt/I_{yy}}u(t)$



Example: Constant rotation

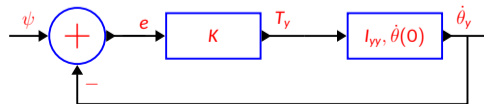
- Only input has changed. $\psi(t) = au(t)$



Example: Constant rotation

- Only input has changed. $\psi(t) = au(t)$

$$\dot{\theta}_y(t) = \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau = \frac{K}{I_{yy}} \int_0^t (\psi(t) - \dot{\theta}_y(t)) d\tau$$

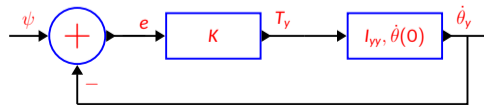


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$$= \frac{K}{I_{yy}} \int_0^t a d\tau - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau = \frac{Kat}{I_{yy}} - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau$$

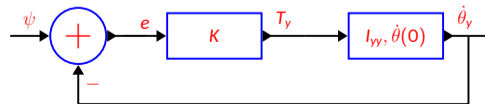


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$$\begin{aligned}\dot{\theta}_y(t) &= \frac{1}{I_{yy}} \int_0^t T_y(\tau) d\tau = \frac{K}{I_{yy}} \int_0^t (\psi(t) - \dot{\theta}_y(t)) d\tau \\ &= \frac{K}{I_{yy}} \int_0^t a d\tau - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau = \frac{Kat}{I_{yy}} - \frac{K}{I_{yy}} \int_0^t \dot{\theta}_y(\tau) d\tau\end{aligned}$$

- $\dot{\theta}_y(t) = au(t)(1 - e^{-Kt/I_{yy}})$



Summary

- We have seen two different kinds of model
- Differential equations - most fundamental way to model
- Acotr model - mostly driven by the software
- Two models are very tightly related