Modeling: Continuous Systems

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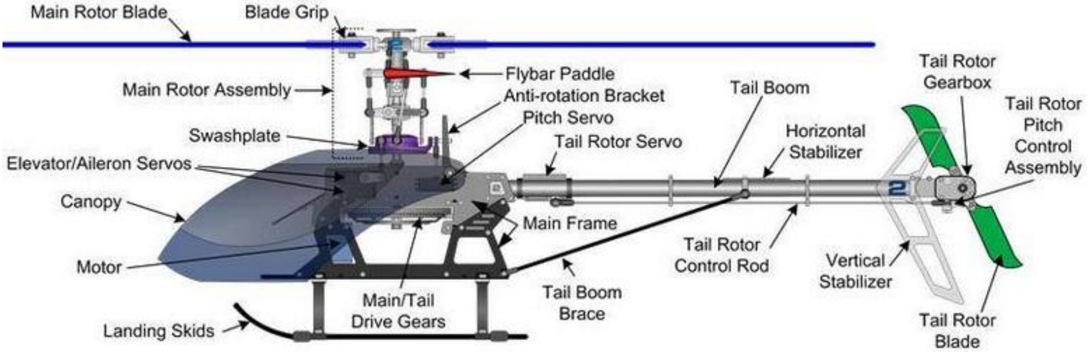
System modeling

- Mimic the real world behavior of the system
- There exist a large variety of systems
  - Mechanical, electrical, chemical, biological, etc.
- Behavior of most of the system can be described using differential equations
- Continuous dynamics
  - Modal models
    - Used for modeling discrete systems
    - For each mode, we have continuous dynamics
- Ordinary differential equation will be used to describe the system
  - Properties like linearity, time invariance, stability, etc. will be considered

Helicopter



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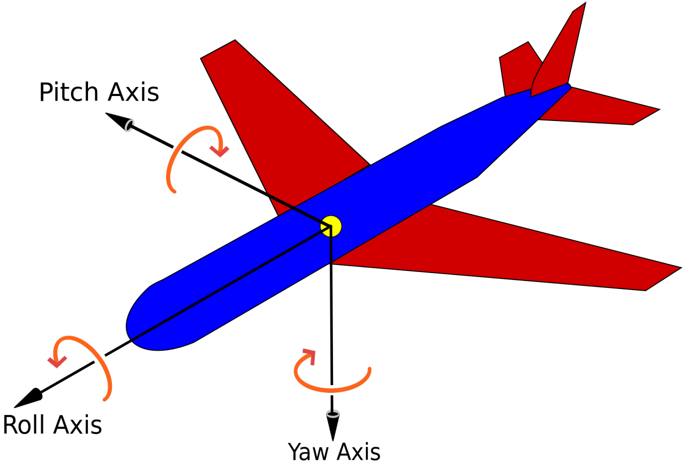

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- Change in position or orientation can be determined by Newton's 2nd law

$$\boldsymbol{F}(t) = M\ddot{\boldsymbol{x}}(t)$$

  - $\boldsymbol{F}$ - force, M - mass and $\ddot{\boldsymbol{x}}$ - second derivative ie. acceleration

Newtonian mechanics (contd.)

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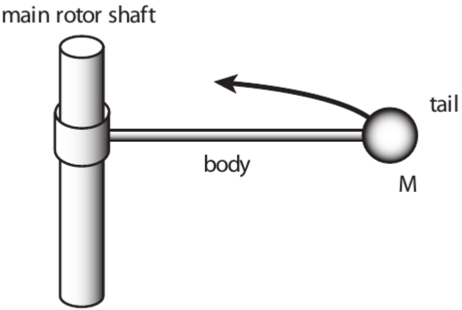
$$\boldsymbol{\theta}(t) \;=\; \boldsymbol{\theta}(0) + t\dot{\boldsymbol{\theta}}(0) + \frac{1}{I} \int_0^t \int_0^\tau \boldsymbol{T}(\alpha)\, d\alpha\, d\tau$$

Helicopter model

- Helicopter has two rotors
  - Main rotor to lift
  - Tail rotor to counter balance spin
- Hence, we have

$$\ddot{\theta}_y(t) = T_y(t)/I_{yy} \Rightarrow$$

$$\dot{\theta}_y(t) = \dot{\theta}_y(0) + \frac{1}{I_{yy}} \int_0^t T_y(\tau)\, d\tau$$



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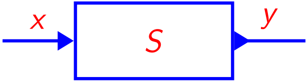
Actor model

- Physical system can be described by input (force, torque) and output (position, orientation, velocity, rotation, etc.)



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  - $S : X \to Y$, $x, y \in \mathbb{R}$

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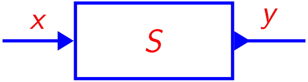

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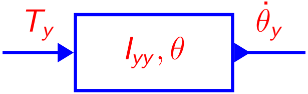
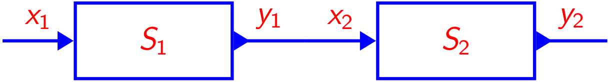


Actor model (contd.)

- Actor models are composable



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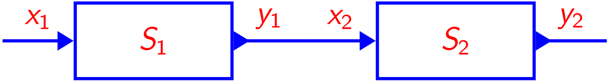

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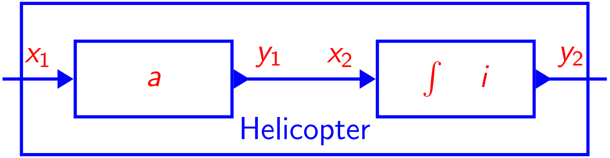
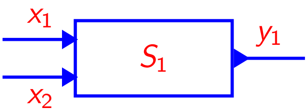


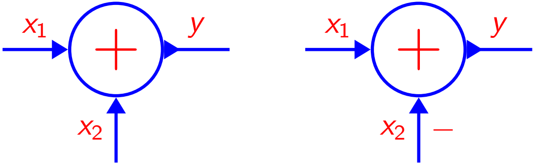
- We have $\forall t \in \mathbb{R} \quad y_2(t) = i + \int_0^t x_2(\tau) d\tau$ where $a = 1/I_{yy}$, $i = \dot{\theta}_y(0)$, $x_1 = T_y$ and $y_2 = \dot{\theta}_y$

Actor model (contd.)

- Actor can have multiple inputs



- Another useful building block is signal adder



  - $y(t) = x_1(t) + x_2(t)$, $y(t) = x_1(t) - x_2(t)$

Properties of systems

- Causal system
- Memoryless systems
- Linear and time invariant
- Stability
- Feedback control

Causal systems

- Output depends only on current and past inputs
- Consider a continuous time signal x
- Let $x|_{t \leq \tau}$ represent restriction in time defined only for $t \leq \tau$
- Consider a continuous time system $S : X \to Y$, the system is causal if for all $x_1, x_2 \in X$ and $\tau \in R$, $\quad x_1|_{t \leq \tau} = x_2|_{t \leq \tau} \Rightarrow S(x_1)|_{t \leq \tau} = S(x_2)|_{t \leq \tau}$
- Strictly causal $\forall \tau \in R$, $\quad x_1|_{t < \tau} = x_2|_{t < \tau} \Rightarrow S(x_1)|_{t \leq \tau} = S(x_2)|_{t \leq \tau}$

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- Example
  - Integrator is strictly causal
  - Adder is not strictly causal but causal
- Strictly causal actors are good for continuous feedback system

Memoryless systems

- A systems has memory if the output depends not only on the current inputs but also on the past inputs
- Formally, $S : X \to Y$ the system is memoryless if there exist a function $f : X \to Y$ such that for all $x \in X, (S(x))(t) = f(x(t))$ for all $t \in R$

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- Example
  - Integrator is not memoryless
  - Adder is memoryless

Linear and time invariant (LTI)

- A systems $S : X \rightarrow Y$ where X and Y are sets of signals is linear if it satisfies the superposition property
  $\forall x_1, x_2 \in X$ and $\forall a, b \in R \quad S(ax_1 + bx_2) = aS(x_1) + bS(x_2)$

- Time invariance means that whether we apply an input to the system now or T seconds from now, the output will be identical except for a time delay of T seconds.
  - Let D_τ be the delay operator such that $(D_\tau(x))(t) = x(t - \tau)$
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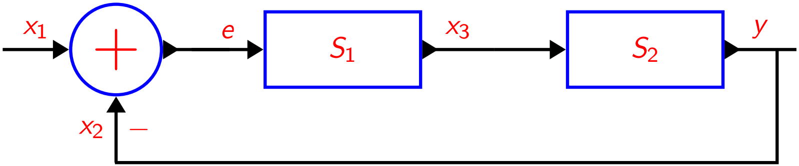
- $\dot{\theta}_y(t) = \frac{1}{I_{yy}} \int_{-\infty}^{t} T_y(\tau) d\tau$ - LTI
- Many systems are approximated to LTI

Stability

- A system is bounded input bounded output stable if the output signal is bounded for all inputs signals that are bounded
- Helicopter is unstable

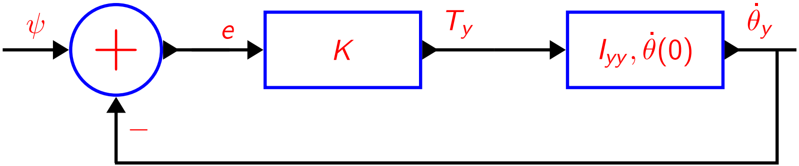
Feedback systems

- A system with feedback has directed cycle where an output from an actor is fed back to affect an input of the same actor



Example: No rotation

- Want to have 0 angular velocity



Example: No rotation (contd.)

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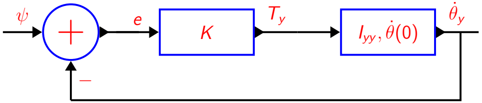

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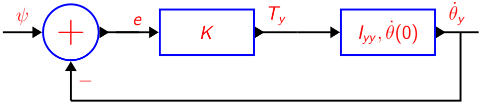

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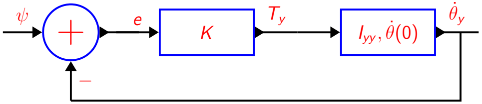

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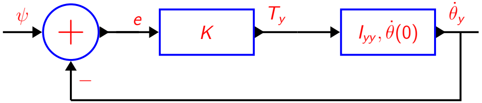

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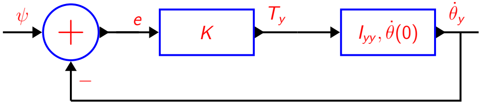

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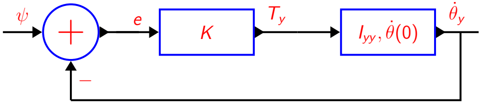
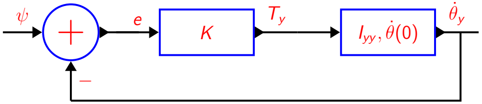
- Therefore we have, $\dot{\theta}_y(t) = \dot{\theta}_y(0)e^{-Kt/I_{yy}}u(t)$



Example: Constant rotation

- Only input has changed. $\psi(t) = au(t)$



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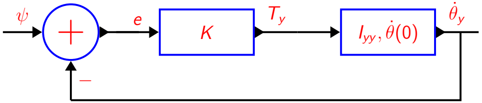

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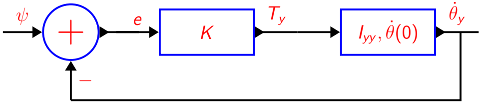

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- $\dot{\theta}_y(t) = au(t)(1 - e^{-Kt/I_{yy}})$