# Introduction to Deep Learning

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# Regularization

# Introduction

- In machine learning, target is to make an algorithm performs well not only on training data but also on new data
- Many strategies exist to reduce test error at the cost of training error
- Any modification we make to a learning algorithm that is intended to reduce its generalization error but not its training error
- Objectives
  - To encode prior knowledge
  - Constraints and penalties are designed to express generic preference for simpler model

# Regularization in DL

- In DL regularization works by trading increased bias for reduced variance
- Consider the following scenario
  - Excluded the true data generating process
    - Underfitting, inducing bias
  - Matched the true data generating process
    - Desired one
  - Included the generating process but also many other generating process
    - Overfitting, variance dominates
  - Goal of regularizer is to take an model overfit zone to desired zone

# Norm penalties

- Most of the regularization approaches are based on limiting the capacity of the model
- Objective function becomes $\tilde{J}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) = J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha\Omega(\boldsymbol{\theta})$
  - $\alpha$ — Hyperparameter denotes relative contribution
  - Minimization of $\tilde{J}$ implies minimization of $J$
  - $\Omega$ penalizes only the weight of affine transform
    - Bias remain unregularized
    - Regularizing bias may lead to underfitting

# $L^2$ parameter regularization

- Weights are closer to origin as $\Omega(\boldsymbol{\theta}) = \frac{1}{2}\|\boldsymbol{w}\|_2^2$
  - Also known as **ridge regression** or **Tikhonov regression**
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$$\boldsymbol{w} = \boldsymbol{w} - \epsilon(\alpha\boldsymbol{w} + \nabla_{\boldsymbol{w}} J(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y}))$$

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- New weights

$$\begin{aligned}\boldsymbol{w} &= \boldsymbol{w} - \epsilon(\alpha\boldsymbol{w} + \nabla_{\boldsymbol{w}} J(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y})) \\ &= \boldsymbol{w}(1 - \epsilon\alpha) - \epsilon\nabla_{\boldsymbol{w}} J(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y})\end{aligned}$$

- Assume quadratic nature of curve in the neighborhood of $\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} J(\boldsymbol{w})$
  - $J(\boldsymbol{w})$ — unregularized cost
  - Perfect scenario for linear regression with MSE

# Jacobian & Hessian

- Derivative of a function having single input and single output — $\frac{dy}{dx}$
- Derivative of function having vector input and vector output that is, $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$
  - Jacobian $\boldsymbol{J} \in \mathbb{R}^{n \times m}$ of $f$ defined as $J_{i,j} = \frac{\partial}{\partial x_j} f(\boldsymbol{x})_i$
- Second derivative is also required sometime
  - For example, $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $\dfrac{\partial^2}{\partial x_i \partial x_j} f$
  - If second derivative is 0, then there is no curvature
- Hessian matrix $\boldsymbol{H}(f)(x)_{ij} = \dfrac{\partial^2}{\partial x_i \partial x_j} f(\boldsymbol{x})$

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  - Jacobian of gradient
  - Symmetric

# Directional derivative

- The directional derivative of a scalar function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ along a vector $\mathbf{v} = (v_1, \ldots, v_n)$ is given by

$$\nabla_{\boldsymbol{v}} f(x) = \lim_{h \to 0} \frac{f(x + h\boldsymbol{v}) - f(x)}{h}$$

- If $f$ is differentiable at point $x$ then

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{v}$$

# Taylor series expansion

- A real valued function differentiable at point $x_0$ can be expressed as

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \cdots .$$

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- When input is a vector

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}^{(0)}) + (\boldsymbol{x} - \boldsymbol{x}^{(0)})^T \boldsymbol{g} + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^{(0)})^T \boldsymbol{H} (\boldsymbol{x} - \boldsymbol{x}^{(0)})$$

  - $\boldsymbol{g}$ — gradient at $\boldsymbol{x}^{(0)}$, $\boldsymbol{H}$ — Hessian at $\boldsymbol{x}^{(0)}$

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  - $\boldsymbol{g}$ — gradient at $\boldsymbol{x}^{(0)}$, $\boldsymbol{H}$ — Hessian at $\boldsymbol{x}^{(0)}$

- If $\epsilon$ is the learning rate, then $f(\boldsymbol{x}^{(0)} - \epsilon \boldsymbol{g}) = f(\boldsymbol{x}^{(0)}) - \epsilon \boldsymbol{g}^T \boldsymbol{g} + \frac{1}{2}\epsilon^2 \boldsymbol{g}^T \boldsymbol{H} \boldsymbol{g}$

# Quadratic approximation

- Let $\boldsymbol{w}^* = \arg\min_{\boldsymbol{w}} J(\boldsymbol{w})$ optimum weights for minimal unregularized cost
- If the objective function is quadratic then $\hat{J}(\boldsymbol{\theta}) = J(\boldsymbol{w}^*) + \frac{1}{2}(\boldsymbol{w} - \boldsymbol{w}^*)^T \boldsymbol{H}(\boldsymbol{w} - \boldsymbol{w}^*)$
  - $\boldsymbol{H}$ is the Hessian matrix of $J$ with respect to $\boldsymbol{w}$ at $\boldsymbol{w}^*$
  - No first order term as $\boldsymbol{w}^*$ is minimum
  - $\boldsymbol{H}$ is positive semidefinite
- Minimum of $\hat{J}$ occurs when $\nabla_{\boldsymbol{w}} \hat{J}(\boldsymbol{w}) = \boldsymbol{H}(\boldsymbol{w} - \boldsymbol{w}^*) = 0$
- With weight decay we have

$$\begin{aligned}
\alpha\tilde{\boldsymbol{w}} + \boldsymbol{H}(\tilde{\boldsymbol{w}} - \boldsymbol{w}^*) &= 0 \\
(\boldsymbol{H} + \alpha\boldsymbol{I})\tilde{\boldsymbol{w}} &= \boldsymbol{H}\boldsymbol{w}^* \\
\tilde{\boldsymbol{w}} &= (\boldsymbol{H} + \alpha\boldsymbol{I})^{-1}\boldsymbol{H}\boldsymbol{w}^*
\end{aligned}$$

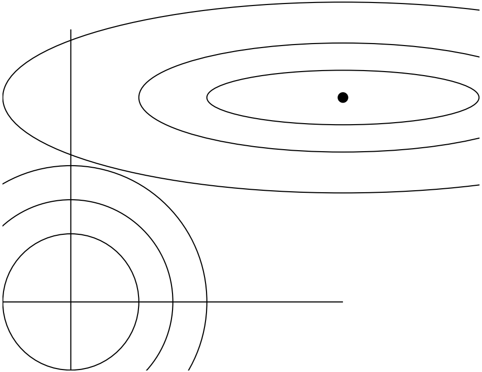
# Quadratic approximation (contd)

- As $\alpha \rightarrow 0$, regularized solution $\hat{\boldsymbol{w}}$ approaches to $\boldsymbol{w}^*$
- As $\alpha \rightarrow \infty$
  - $\boldsymbol{H}$ is symmetric, therefore $\boldsymbol{H} = \boldsymbol{Q\Lambda Q}^T$. Now we have

$$\begin{aligned}
\tilde{\boldsymbol{w}} &= (\boldsymbol{Q\Lambda Q} + \alpha \boldsymbol{I})^{-1}\boldsymbol{Q\Lambda Q}^T\boldsymbol{w}^* \\
&= \left[\boldsymbol{Q}(\boldsymbol{\Lambda} + \alpha \boldsymbol{I})\boldsymbol{Q}^T\right]^{-1}\boldsymbol{Q\Lambda Q}^T\boldsymbol{w}^* \\
&= \boldsymbol{Q}(\boldsymbol{\Lambda} + \alpha \boldsymbol{I})^{-1}\boldsymbol{\Lambda Q}^T\boldsymbol{w}^*
\end{aligned}$$

  - Weight decay rescale $\boldsymbol{w}^*$ along the eigen vector of $\boldsymbol{H}$
    - Component of $\boldsymbol{w}^*$ that is aligned to i-th eigen vector, will be rescaled by a factor of $\frac{\lambda_i}{\lambda_i+\alpha}$
    - $\lambda_i \gg \alpha$ — regularization effect is small

# $L^2$ Norm



# Linear regression

- For linear regression cost function is $(\boldsymbol{X}\boldsymbol{w}-\boldsymbol{y})^T(\boldsymbol{X}\boldsymbol{w}-\boldsymbol{y})$
- Using $L^2$ regularization we have $(\boldsymbol{X}\boldsymbol{w}-\boldsymbol{y})^T(\boldsymbol{X}\boldsymbol{w}-\boldsymbol{y})+\frac{1}{2}\alpha\boldsymbol{w}^T\boldsymbol{w}$

# Linear regression

- For linear regression cost function is $(\boldsymbol{Xw} - \boldsymbol{y})^T(\boldsymbol{Xw} - \boldsymbol{y})$
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- Solution for normal equation $\boldsymbol{w} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{y}$

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- Solution for with weight decay $\boldsymbol{w} = (\boldsymbol{X}^T\boldsymbol{X} + \alpha\boldsymbol{I})^{-1}\boldsymbol{X}^T\boldsymbol{y}$

# $L^1$ regularization

- Formally it is defined as $\Omega(\boldsymbol{\theta}) = \|\boldsymbol{w}\|_1 = \sum_i |w_i|$
- Regularized objective function will be $\tilde{J}(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y}) = \alpha\|\boldsymbol{w}\|_1 + J(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y})$

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- Regularized objective function will be $\tilde{J}(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y}) = \alpha\|\boldsymbol{w}\|_1 + J(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y})$
- The gradient will be $\nabla_{\boldsymbol{w}}\tilde{J}(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y}) = \alpha \text{sign}(\boldsymbol{w}) + \nabla_{\boldsymbol{w}} J(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y})$
  - Gradient does not scale linearly compared to $L^2$ regularization
- Taylor series exapansion with approximation provides $\nabla_{\boldsymbol{w}}\hat{J}(\boldsymbol{w}) = \boldsymbol{H}(\boldsymbol{w} - \boldsymbol{w}^*)$
- Simplification can be made by assuming $\boldsymbol{H}$ to be diagonal
  - Apply PCA on the input dataset

# $L^1$ regularization

- Quadratic approximation of $L^1$ regularization objective function becomes $\hat{J}(\boldsymbol{w}; \boldsymbol{X}, \boldsymbol{y}) = J((\boldsymbol{w}^*; \boldsymbol{X}, \boldsymbol{y}) + \sum_i \left[\frac{1}{2} H_{i,i}(\boldsymbol{w}_i - \boldsymbol{w}_i^*)^2 + \alpha|w_i|\right]$
- So, analytical in each dimension will be $w_i = \text{sign}(w_i^*) \max\left\{|w_i^*| - \frac{\alpha}{H_{i,i}}, 0\right\}$
- Consider the situation when $w_i^* > 0$
  - If $w_i^* \leq \frac{\alpha}{H_{i,i}}$, optimal value for $w_i$ will be 0 under regularization
  - If $w_i^* > \frac{\alpha}{H_{i,i}}$, $w_i$ moves towards 0 with a distance equal to $\frac{\alpha}{H_{i,i}}$

# Constrained optimization

- Cost function regularized by norm penalty is given by

$$\tilde{J}(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) = J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha\Omega(\boldsymbol{\theta})$$

- Let us assume $f(x)$ needs to be optimized under a set of equality constraints $g^{(i)}(x) = 0$ and inequality constraints $h^{(j)}(x) \leq 0$, then generalized Lagrangian is then defined as

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\alpha}) = f(\boldsymbol{x}) + \sum_i \lambda_i g^{(i)}(x) + \sum_j \alpha_i g^{(i)}(x)$$

- If there exists a solution then

$$\min_x \max_{\boldsymbol{\lambda}} \max_{\boldsymbol{\alpha} \geq 0} = L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\alpha}) = \min_x f(\boldsymbol{x})$$

  - This can be solved by $\nabla_{\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\alpha}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\alpha}) = 0$

# Constraint optimization (contd.)

- Suppose $\Omega(\boldsymbol{\theta}) < k$ needs to be satisfied. Then regularization equation becomes

$$L(\boldsymbol{\theta}, \alpha; \boldsymbol{X}, \boldsymbol{y}) = J(\boldsymbol{\theta}; \boldsymbol{X}, \boldsymbol{y}) + \alpha(\Omega(\boldsymbol{\theta}) - k)$$

  - Solution to the constrained problem

$$\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \max_{\alpha > 0} L(\boldsymbol{\theta}, \alpha)$$

# Dataset augmentation

- If data are limited, fake data can be added to training set
  - Computer vision problem
  - Speech recognition
- Easiest for classification problem
- Very effective in object recognition problem
  - Translating
  - Rotating
  - Scaling
    - Need to be careful for 'b' and 'd' or '6' and '9'
- Injecting noise to input data can be viewed as data augmentation

# Multitask learning

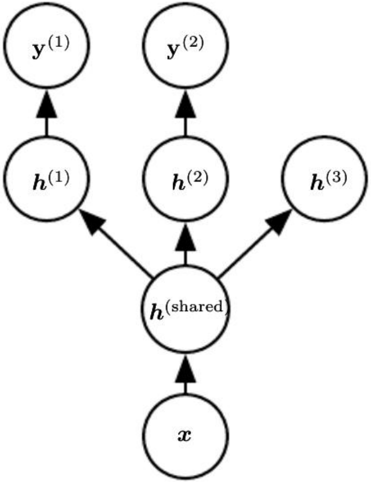
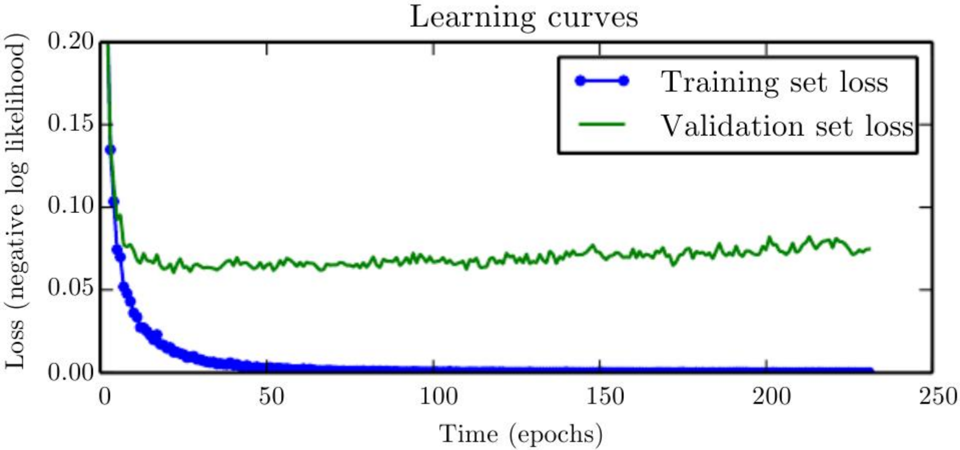



Image source: Deep Learning Book

# Early stopping

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# Early stopping approach

- Intialize the parameters
- Run training algorithm for $n$ steps and update $i = i + n$
- Compute error on the validation set ($v'$)
- If $v'$ is less than previous best, then update the same. Start step 2 again
- If $v'$ is more than the previous best, then increament the count that stores the number of such occurrences. If the count is less than a threshold go to step 2, otherwise exit.

# Early stopping (contd)

- Number of tarining step is a hyperparameter
  - Most hyperparameters that control model capacity have U-shaped curve
- Additional cost for this approach is to store the parameters
- Requires a validation set
  - It will have two passes
    - First pass uses only training data for update of the parameters
    - Second pass uses both training and validation data for update of the parameters
  - Possible strategies
    - Intialize the model again, retrain on all data, train for the same number of steps as obtained by early stopping in pass 1
    - Keep the parameters obtained from the first round, continue training using all data until the validation loss is than the training loss at the early stopping point
- It reduces computational cost as it limits the number of iteration
- Provides regularization without any penalty

# Early stopping as regularizer

- Let us assume $\tau$ training iteration, $\epsilon$ learning rate
  - $\epsilon\tau$ — measures effective capacity
- We have, $\hat{J}(\boldsymbol{\theta}) = J(\boldsymbol{w}^*) + \frac{1}{2}(\boldsymbol{w} - \boldsymbol{w}^*)\boldsymbol{H}(\boldsymbol{w} - \boldsymbol{w}^*)$ and $\nabla_{\boldsymbol{w}}\hat{J}(\boldsymbol{w}) = \boldsymbol{H}(\boldsymbol{w} - \boldsymbol{w}^*)$
- Assume $\boldsymbol{w}^{(0)} = 0$
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\boldsymbol{Q}^T\boldsymbol{w}^{(\tau)} &= [\boldsymbol{I} - (\boldsymbol{I} - \epsilon\boldsymbol{\Lambda})^\tau]\boldsymbol{Q}^T\boldsymbol{w}^*
\end{aligned}
$$

# Early stopping as regularizer (contd)

- Assuming $\boldsymbol{w}^{(0)} = 0$ and $\epsilon$ is small value such that $|1 - \epsilon\lambda_i| < 1$
- From $L^2$ regularization, we have

$$\begin{aligned}\boldsymbol{Q}^T\tilde{\boldsymbol{w}} &= (\boldsymbol{\Lambda} + \alpha\boldsymbol{I})^{-1}\boldsymbol{\Lambda}\boldsymbol{Q}^T\boldsymbol{w}^* \\ \boldsymbol{Q}^T\tilde{\boldsymbol{w}} &= [\boldsymbol{I} - (\boldsymbol{\Lambda} + \alpha\boldsymbol{I})^{-1}\alpha]\boldsymbol{Q}^T\boldsymbol{w}^*\end{aligned}$$

- Therefore we have, $(\boldsymbol{I} - \epsilon\boldsymbol{\Lambda})^{\tau} = (\boldsymbol{\Lambda} + \alpha\boldsymbol{I})^{-1}\alpha$
- Hence, $\tau \approx \frac{1}{\epsilon\alpha}$ $\alpha \approx \frac{1}{\tau\epsilon}$

# Bagging

- Also known as Bootstrap aggregating
- Reduces generalization error by combining several models
- Train multiple models then vote on output for the test example
  - Also known as model averaging, ensemble method
- Suppose we have $k$ regression model and each model makes an error $\epsilon_i$ such that $\mathbb{E}(\epsilon_i) = 0$, $\mathbb{E}(\epsilon_i^2) = v$, $\mathbb{E}(\epsilon_i \epsilon_j) = c$
- Error made by average prediction $\frac{1}{k}\sum_i \epsilon_i$
- Expected mean square error

$$\mathbb{E}\left[\left(\frac{1}{k}\sum_i \epsilon_i\right)^2\right] = \frac{1}{k^2}\mathbb{E}\left[\sum_i \left(\epsilon_i^2 + \sum_{i \neq j} \epsilon_i \epsilon_j\right)\right] = \frac{v}{k} + \frac{k-1}{k}c$$

# Dropout

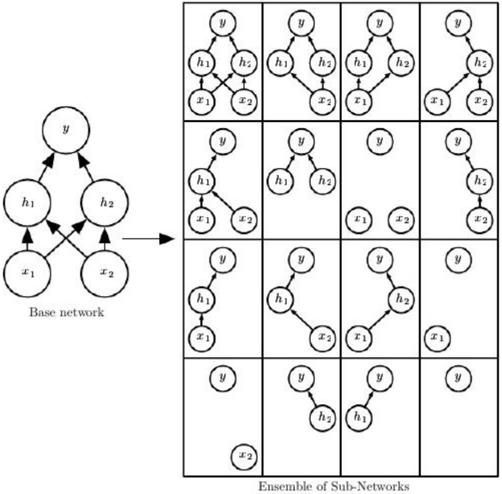


Image source: Deep Learning Book

# Dropout

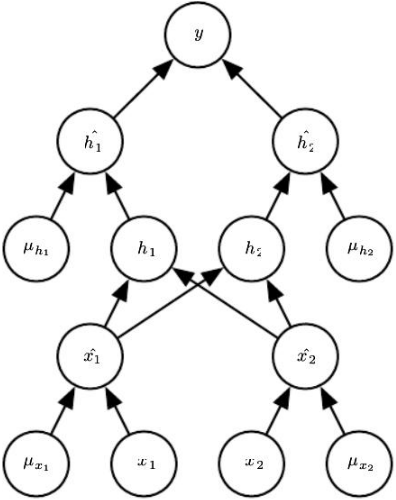


Image source: Deep Learning Book

# Adversarial training

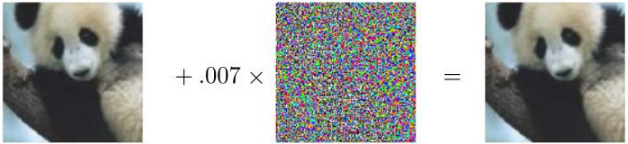


Image source: Deep Learning Book